Solutions Practice Midterm 2

Part I

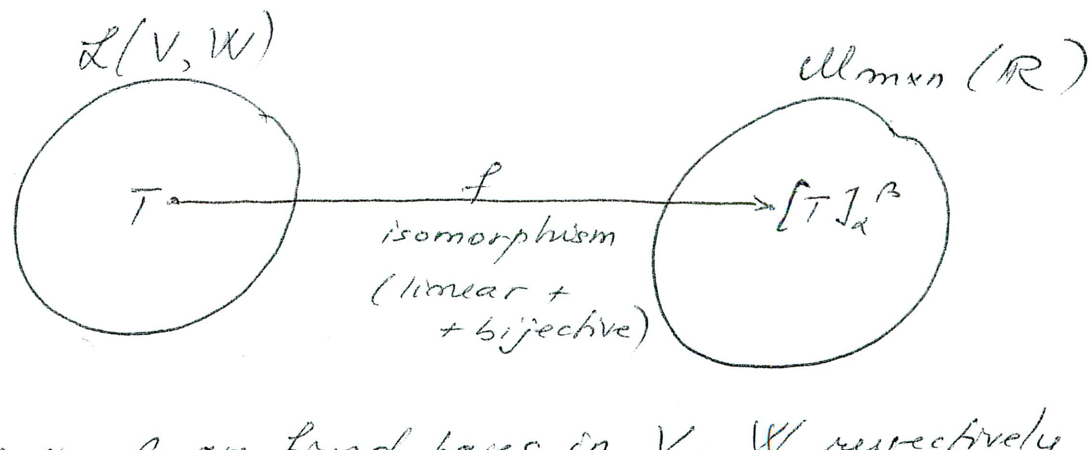
① $[T]_\alpha^\beta = \left([T(1)]_\beta \; [T(x)]_\beta \; [T(x^2)]_\beta\right) =$

$$= \left(\left[\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right]_\beta \; \left[\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\right]_\beta \; \left[\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}\right]_\beta \right) = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Correct answer: d

② By definition, the fact that $(U+T)(x+y) = (U+T)(x) + (U+T)(y)$ represents the good behavior of $U+T$ w.r.t. addition. So the correct answer is c.

③ We have: $\underbrace{\mathcal{L}(V, W)}_{\dim n \;\; \dim m} \simeq \mathcal{M}_{m\times n}(\mathbb{R})$,



where α, β are fixed bases in V, W respectively.

(2)

By definition, the good behavior of f w.r.t. addition is the fact that $f(U+T) = f(U) + f(T)$ for all $U, T \in \mathcal{L}(V, W)$. But this means exactly that $[U+T]_\alpha^\beta = [U]_\alpha^\beta + [T]_\alpha^\beta$, so the correct answer is (c).

(4)

1) True for all functions T, in particular for linear transformations

2) True for all functions T, in particular for linear transformation.

3) True.

T invertible $\Rightarrow$ T bijective $\Rightarrow$ T surjective $\Rightarrow$

$\Rightarrow \operatorname{Im} T = W \Rightarrow \dim(\operatorname{Im} T) = \dim(W)$

T^{-1} invertible $\xRightarrow{\text{similarly}} \dim(\operatorname{Im} T^{-1}) = \dim(V)$.

But $\dim(V) = \dim(W)$ (as T isomorphism), hence $\dim(\operatorname{Im} T) = \dim(\operatorname{Im} T^{-1})$, which means $\operatorname{rank}(T) = \operatorname{rank}(T^{-1})$.

4) True.

T invertible + linear $\Rightarrow T^{-1}$ linear $\Rightarrow$

$\Rightarrow T^{-1}$ behaves well w.r.t. scalar multiplication $\Rightarrow$

$\Rightarrow T^{-1}(ax) = a T^{-1}(x)$ all $a \in \mathbb{R}$ and all $x \in V$.

Correct answer: (a)

5

1) True, it's an important property of an isomorphism

2) True, it's an important property of an isomorphism as well.

3) As $\{\vec{v_1}, \dots, \vec{v_n}\}$ basis for V, $\dim(V) = n$.
As $\dim(V) = \dim(W)$, we'll have also $\dim(W) = n$.
As any vector space of dim n is isomorphic to $\mathbb{R}^n$, we'll have $W \cong \mathbb{R}^n$. So 3) is true.

4) Let $\{\vec{v_1}, \dots, \vec{v_n}\}$ be linearly independent in V. Let $c_1, \dots, c_n \in \mathbb{R}$ be s.t. $c_1 T(\vec{v_1}) + \dots + c_n T(\vec{v_n}) = \vec{0_W}$.
By linearity $\Rightarrow T(c_1\vec{v_1} + \dots + c_n\vec{v_n}) = \vec{0_W}$. By the injectivity of $T \Rightarrow c_1\vec{v_1} + \dots + c_n\vec{v_n} = \vec{0_V}$. By the linearly independence of $\vec{v_1}, \dots, \vec{v_n} \Rightarrow c_1 = \dots = c_n = 0$.
So $\{T(\vec{v_1}), \dots, T(\vec{v_n})\}$ linearly independent in W.
Thus 4) is true.

5) Let $\{\vec{v_1}, \dots, \vec{v_n}\}$ be a generating set for V and let $\vec{w} \in W$. By surjectivity, $\exists\, \vec{v} \in V$ s.t. $T(\vec{v}) = \vec{w}$.
$\{\vec{v_1}, \dots, \vec{v_n}\}$ generating V, we can write $\vec{v} = c_1\vec{v_1} + \dots + c_n\vec{v_n}$, and we'll have $T(c_1\vec{v_1} + \dots + c_n\vec{v_n}) = \vec{w}$. This means $c_1 T(\vec{v_1}) + \dots + c_n T(\vec{v_n}) = \vec{w}$, so $\vec{w}$ can be written as a linear combination of $T(\vec{v_1}), \dots, T(\vec{v_n})$.
Thus $\{T(\vec{v_1}), \dots, T(\vec{v_n})\}$ generating set for W. So 5) true

Correct answer : a

6

1) True, because T bijective $\Rightarrow$ T invertible and the invertibility of T is equivalent to the invertibility of the matrix $[T]_\alpha^\beta$ (for any bases α and β).

2) True, because the invertibility of $[T]_\alpha^\beta$ is equivalent to the invertibility of T and the invertibility of T implies through the dimension theorem that $\operatorname{rank}(T) = \dim(V)$

3) True according to 2) and to the fact that $\dim(V) = \dim(W)$, T being an isomorphism (linear + bijective).

4) True, as $T^{-1}T = 1_V$, so $[T^{-1}T]_\alpha^\alpha = [1_V]_\alpha^\alpha = I_n$, where $n = \dim V$.

5) Not true, as $[1_W]_\beta^\beta = I_m$ and $[1_V]_\alpha^\alpha = I_n$, where $m = \dim(W)$ and $n = \dim(V)$. (True only when $\dim(W) = \dim(V)$).

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "T^{-1}T = 1_V"'] & W \arrow[r, "T^{-1}"] & V \end{tikzcd}$$

Correct answer : b

7 $Q \overset{\text{def}}{=} [1_{P_2(\mathbb{R})}]_\alpha^\beta = \left([1_{P_2(\mathbb{R})}(1)]_\beta \;\; [1_{P_2(\mathbb{R})}(x)]_\beta \;\; [1_{P_2(\mathbb{R})}(x^2)]_\beta \right)$

$= \left([1]_\beta \;\; [x]_\beta \;\; [x^2]_\beta \right) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix}$, as

$1 = c_1(2x^2 - x) + c_2(3x^2 + 1) + c_3 x^2 \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$

$x = c_1(2x^2 - x) + c_2(3x^2 + 1) + c_3 x^2 \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$x^2 = c_1(2x^2 - x) + c_2(3x^2 + 1) + c_3 x^2 \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Correct answer: b

8 Choose $\alpha = \{1, x\}$. Then $[T]_\alpha = \left([T(1)]_\alpha \; [T(x)]_\alpha \right) =$
$= \left([2+x]_\alpha \; [x]_\alpha \right) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

$p_{[T]_\alpha}(\lambda) = \lambda^2 - T \cdot \lambda + D = \lambda^2 - 3\lambda + 2 \; \begin{matrix} \nearrow 2 \\ \searrow 1 \end{matrix}$ (eigenvalues)

Correct answer: a

9 Recall the formula $[T(\vec{v})]_\beta = [T]_\alpha^\beta \cdot [\vec{v}]_\alpha$ for all $\vec{v} \in V$
($V = \mathbb{R}^2$ in our case) and let's apply it for T^{-1}:

$$[T^{-1}(\vec{v})]_\alpha = [T^{-1}]_\beta^\alpha \cdot [\vec{v}]_\beta$$

for all $\vec{v} \in W$ ($W = \mathbb{R}^2$ in our case). As $[T^{-1}]_\beta^\alpha = \left([T]_\alpha^\beta\right)^{-1}$,
we'll have $[T^{-1}]_\beta^\alpha = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and applying the last
formula for $\vec{v} \in \mathbb{R}$ s.t. $[\vec{v}]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ we get:

$$[T^{-1}(\vec{v})]_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Correct answer: c.

10 The proof of the specified theorem looks like this:

$$K = \{x \in \mathbb{R}^n \,/\, Ax = 0\} \overset{\text{def}}{\underset{L_A}{=}} \{x \in \mathbb{R}^n \,/\, L_A(x) = 0\} \overset{\text{def}}{\underset{\text{kernel}}{=}}$$

$= \ker(L_A)$.

Dimension theorem $\Rightarrow \dim(\ker(L_A)) + \operatorname{rank}(L_A) = \dim(\mathbb{R}^n)$

$$\underset{\dim(K)}{\parallel} \qquad \underset{\operatorname{rank}(A)}{\parallel} \qquad \underset{n}{\parallel}$$

$$\Downarrow$$

$$\dim(K) = n - \operatorname{rank}(A).$$

So it uses all 1), 2), 3).

Correct answer: a

11 $p_A(\lambda) = \lambda^2 - T\cdot\lambda + D = \lambda^2 - (-1)\lambda - 2 = (\lambda+2)(\lambda-1)$

$$-2 \qquad 1$$

For $\lambda_1 = 1 \longrightarrow$ eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = -2 \longrightarrow$ eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

As $\dim(E_i) = m_{\lambda_i}$ for each λ_i, A is diagonalizable, with $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$. Thus $A = QDQ^{-1}$,

hence $A^n = \underbrace{(QDQ^{-1})(QDQ^{-1})\dots(QDQ^{-1})}_{n\text{ times}} = QD^nQ^{-1}$.

$$= \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (-2)^n & 0 \\ -(-2)^n + 1 & 1 \end{pmatrix}.$$

Correct answer: a.

II.

12 True. Because both are isomorphic to $\mathbb{R}^n$ (supposing the dim. of both of them is n).

13 True. See ex. 10/2.4 (A "one-sided" inverse is a "two-sided" inverse.)

14 True. See ex. 9/2.4 as well as the lemma before th. 3.2 / section 3.1.

15 $T(x) = (-1) + 1\cdot(1+x) = x$, so x is eigenvector for T corresponding to the eigenvalue 1.

$T(1-x) = 2 + 0\cdot(1+x) = 2 \neq \lambda(1-x)$ all $\lambda \in \mathbb{R}$.

So $1-x$ is not eigenvector for T. So the answer is: False.

16 Choose $\alpha = \{1, x\}$. Then $[T]_\alpha = \left([T(1)]_\alpha \ [T(x)]_\alpha\right) =$

$$= \left([2+x]_\alpha \ [x]_\alpha\right) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$p_{[T]_\alpha}(\lambda) = \lambda^2 - T\cdot\lambda + D = \lambda^2 - 3\lambda + 2 \begin{matrix} 1 \\ 2 \end{matrix}.$$

As all eigenvalues are real and distinct the condition $\dim(E_{\lambda_i}) = m_{\lambda_i}$ is satisfied automatically, so $[T]_\alpha$ and hence T are diagonalizable.

Correct answer: True.

17 This is true. It goes not only for the real number 2, but for any real number λ. To prove it we'll use the fact that two vectors in V are equal iff their coordinate vectors in a certain basis are equal. We'll have:

$$\lambda \text{ eigenvalue of } T \text{ corresponding to } \vec{v} \in V,\ \vec{v} \neq \vec{0}$$

$$\Updownarrow$$

$$T(\vec{v}) = \lambda\vec{v}$$

$$\Updownarrow$$

$$[T(\vec{v})]_\alpha = [\lambda\vec{v}]_\alpha$$

$$\Updownarrow \text{ th. 2.14}$$

$$[T]_\alpha[\vec{v}]_\alpha = \lambda[\vec{v}]_\alpha$$

$$\Updownarrow$$

$$\lambda \text{ eigenvalue of } [T]_\alpha \text{ corresponding to } [\vec{v}]_\alpha \in R^n.$$

18 False.

Counterexample: The solution set of the system $x_1 + x_2 = 5$ (which means $x_2 = 5 - x_1$) is $K = \left\{ \begin{pmatrix} a \\ 5-a \end{pmatrix}, a \in \mathbb{R} \right\}$, which is not a subspace of $\mathbb{R}^2$ (it does not contain the zero vector).

19 It's true.

Suppose A can be obtained from I_n by 2 row operations and 3 column operations (goes similarly for m instead of 2 and p instead of 3). Then:

$$A = \underbrace{(E_1 E_2) I_n (G_1 G_2 G_3)}_{\text{elementary matrices}} \quad (1)$$

$$\Downarrow$$

$$A^{-1} = (G_3^{-1} G_2^{-1} G_1^{-1}) I_n (E_2^{-1} E_1^{-1})$$

$$\Downarrow$$

$$I_n = (G_1 G_2 G_3) A^{-1} (E_1 E_2) \quad (2)$$

(1) and (2) $\Rightarrow A = (E_1 E_2)[(G_1 G_2 G_3) A^{-1} (E_1 E_2)](G_1 G_2 G_3)$,

so A can be obtained from A^{-1} by a finite # of elementary operations.

(20) True. Neither the multiplication by an elementary matrix, nor the multiplication by a general invertible matrix will change the rank of A. (See th. 3.4/3.2 and its corollary).

(21) True.
The theorem involved is 3.4 a). Its proof uses that Q is invertible, so L_Q is invertible, hence bijective (injective + surjective) and the surjectivity is used in saying that

$$L_Q(\mathbb{R}^n) = \mathbb{R}^n$$

(the image of L_Q is the whole $\mathbb{R}^n$).

(22) True.
The proof goes like this:

$$\dim(\text{Column space } A) \underset{3.5}{\overset{th.}{=}} \text{rank}(A) = \text{rank}(A^t) \underset{3.5}{\overset{th.}{=}}$$

$$= \dim(\text{Column space } A^t) \underset{A^t}{\overset{\text{def}}{=}} \dim(\text{Row space } A). \ \square$$

23 True.

Because the maximum # of independent columns of A =
= the maximum # of independent rows of A and both
these numbers equal the rank(A) (Corollary 2 of th. 3.6 / 3.2)

24 $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix}$ and $(A/b) = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 3 & 2 \end{pmatrix}$

rank(A) = rank$(A/b) = 2$, so (th. 3.11) the system
is consistent.

Answer: True.

25 False.

Its coefficient matrix A is invertible iff the system has
unique solution (th. 3.10), which is not equivalent
to the system having at least a solution.

26 False.

Counterexample: $\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \end{cases}$ $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ has
strictly less than two independent rows, and still the

system is consistent.

27) True (Th. 5.1)

28) True.

As $\exists$ three linearly independent eigenvectors for λ_1, we have:

$$3 \le \dim(E_1) \underset{5\text{-}7}{\overset{th.}{\le}} m_1 = 3,$$

so $\dim(E_1) = m_1$. As for the eigenvalues λ_2 and λ_3, whose multiplicity is 1, the condition $\dim(E_{\lambda_i}) = m_{\lambda_i}$ is automatically satisfied, we have $\dim(E_{\lambda_i}) = m_{\lambda_i}$ for all the eigenvalues, so T is diagonalizable.

29) A lower-triangular $\Rightarrow$ eigenvalues A = elements on the main diagonal $\begin{cases} 4 \text{ (multiplicity 2)} \\ 5 \text{ (-//- 1)}. \end{cases}$

For $\lambda_1 = 4$: $\begin{pmatrix} 4-4 & 0 & 0 \\ 1 & 4-4 & 0 \\ 0 & 0 & 5-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_3 = 0$

$\Rightarrow E_4 = \left\{ a \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, a \in \mathbb{R} \right\} \Rightarrow \dim(E_4) = 1 < m_4 = 2.$

The condition for diagonalizability fails to be satisfied, so A is not diagonalizable.

Answer: False.

(30) False.

$\{\vec{v_1}, \dots, \vec{v_n}\}$ basis for V consisting in eigenvectors of T iff $\{[\vec{v_1}]_\alpha, \dots, [\vec{v_n}]_\alpha\}$ basis for $\mathbb{R}^n$ consisting in eigenvectors of $[T]_\alpha$ (not an arbitrary basis of $\mathbb{R}^n$).

Counterexp.: $\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$, but $\{x, 1-x\}$

$$\begin{pmatrix}0\\1\end{pmatrix} = [x]_\alpha, \quad \begin{pmatrix}1\\-1\end{pmatrix} = [1-x]_\alpha$$

α canonical basis of $P_1(\mathbb{R})$

does not represent a basis for $P_1(\mathbb{R})$ consisting in eigenvectors of T (T is the operator in ex. 15)

(31) True.

If $\vec{v}$ is an eigenvector of T corresponding to the eigenvalue λ, then $\vec{v} \neq \vec{0}$ and $T(\vec{v}) = \lambda \vec{v}$, hence $T(T(\vec{v})) = T(\lambda \vec{v}) = \lambda T(\vec{v}) = \lambda(\lambda \vec{v}) = \lambda^2 \vec{v}$, so $T^2(\vec{v}) = \lambda^2 \vec{v}$, which means that $\vec{v}$ is an eigenvector for T^2 corresponding to the eigenvalue λ^2.

32 True.

$\{1+x, 1-x\}$ is a basis of $P_1(\mathbb{R})$ because it is linearly independent and contains exactly 2 vectors ($\dim P_1(\mathbb{R}) = 2$).

Moreover, each of these vectors is an eigenvector for T:

$T(1+x) = 3(1+x)$, so $1+x$ is eigenvector corresponding to the eigenvalue 3

$T(1-x) = 1-x$, so $1-x$ is eigenvector corresponding to the eigenvalue 1.

33 True (th. 5.1)

As $P_1(\mathbb{R})$ has a basis consisting only in eigenvectors of T, T will be diagonalizable.