Solutions Midterm 1

① False.

$0 \cdot x = 0 \cdot y \not\Rightarrow x = y$.

② . False.

$\begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} \in W$, but $4 \cdot \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} \notin W$, so W is not closed to scalar multiplication.

(Other motivation: $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin W$).

③ True.

- $W \neq \emptyset$
- W closed to addition ($x, y \in W \Rightarrow 1 \cdot x + y \in W$)
- W closed to scalar multiplication (W contains 0 as for $x \in W$ we have $(-1) \cdot x + x \in W$; then for all $a \in \mathbb{R}$ and $x \in W$ we have $a \cdot x + 0 \in W$).

④ False.

$f(x) = x^2$ is even, so $f \in W$.

$g(x) = x$ is odd, so $g \in W$.

But $f + g$ is neither odd, nor even, so $\notin W$.

5. True.

From $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ we get $a = d = 0$ and $c = -b$.

So $W = \left\{ A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},\ b \in \mathbb{R} \right\}$ = subspace of dim 1 of $M_2(\mathbb{R})$.

depends on one parameter

6. True.

The hypothesis $\Rightarrow$ S is linearly independent.
Also, S is a generating set for $W = \text{span}(S)$, so S it is a basis for W. Then, by the theorem 1-8, every vector in W can be uniquely written as a linear combination of vectors in S.

7. True.

$$-x^3 + 2x^2 + 3x + 3 = (-1)(x^3 + x^2 + x + 1) + 3(x^2 + x + 1) + 1 \cdot (x + 1)$$

8. False.

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$ spans $\mathbb{R}^3$, as the corresponding determinant is nonzero, and $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ 1 \end{pmatrix} \right\}$ also spans $\mathbb{R}^3$, because adding the vector $\begin{pmatrix} 10 \\ 0 \\ 1 \end{pmatrix}$ does not destroy the quality of the initial set of being a generating set.

9 True.

We already saw that $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\3\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\2\end{pmatrix}\right\}$ spans $\mathbb{R}^3$, so it is linearly independent. Adding the 4^{th} vector we exceed the dim of $\mathbb{R}^3$, so the resulting set can no longer be linearly independent.

10 False.

To get a basis for $\mathbb{R}^3$ we would need three vectors from the four given. But any three we choose they have a zero determinant, so they can not represent a basis.

11 True.

Add for instance $\begin{pmatrix}0\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and you'll get a basis.

12 . True

Hypothesis gives that $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent. But this is equivalent to $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ being a generating set for $\mathbb{R}^3$.

13 False.

The set contains five vectors, so it exceeds the dim of $P_3(\mathbb{R})$, which is four.

14. True.

As $\dim V = 2$ and $\{\vec{u}+\vec{v}, 3\vec{u}\}$ contains exactly two vectors, it's enough to check only the linear independence.

If $c_1(\vec{u}+\vec{v}) + c_2(3\vec{u}) = \vec{0_V}$, then $(c_1+3c_2)\vec{u} + c_1\vec{v} = \vec{0_V}$, so, by the linear independence of $\vec{u}$ and $\vec{v}$, c_1+3c_2 and c_1 will both be zero. Hence $c_1 = c_2 = 0$.

15. True.

$W = \left\{ a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\-1\\1\end{pmatrix}, a,b \in \mathbb{R} \right\}$, so a basis for W is $\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix} \right\}$. By multiplying the first vector with 2 and the second with 4 we still get a basis.

16. True.

In fact $W = \{ f \in P_2(\mathbb{R}) \mid f \text{ divizible by } (x-3) \} =$

$= \{ \underbrace{(x-3)(ax+b)}_{\text{depends on 2 parameters}} \mid a,b \in \mathbb{R} \}$ = subspace of dim 2 of $P_2(\mathbb{R})$.

17 . True.

If you put together a basis for W_1 with a basis for W_2 you get a basis for V. So:

# of elements in a basis for V =

= (# of elements in a basis for W_1) + (# of elements in a basis for W_2).

18 True.

See the corollaries after the Replacement Th. (section 1.6)

19 True.

See the same corollaries.

20 . True.

W is a subspace of the arrival space. T is a linear transformation, so the preimage of a subspace will be a subspace in the departure space. (see ex. 20 / section 2.1)

21 . False.

T is not injective:

$x^3 - 2x \neq x^3 - 2x + 5$, but $(x^3-2x)' = (x^3-2x+5)'$

22. True.

The theorem says: V, W have finite and equal dim.

$$\Downarrow$$

T injective $\Leftrightarrow$ T surjective $\Leftrightarrow$ $\text{Rank}(T) = \dim(V)$.

So we have:

$\ker(T) = \{\vec{0}\} \Leftrightarrow T$ injective $\Leftrightarrow \text{Rank}(T) = \dim(V) \underset{\text{are equal}}{\overset{\text{dim}}{\Leftrightarrow}} \text{Rank}(T) = \dim(W)$.

23. True.

T surjective would mean $\text{Im}\, T = W$, so $\text{Rank}(T) = \dim W$, so, from the dimension th., $\text{Nullity}(T) + \dim W = \dim V$. Contradiction with $\dim V < \dim W$.

24. True.

$\text{Im}\, T \subseteq W$, so $\dim(\text{Im}\, T) \leq \dim W$, which means $\text{Rank}(T) \leq \dim W$.

Also, by dimension th., $\text{Rank}(T) \leq \dim V$.

25. True.

Between two spaces of finite and equal dimension, injectivity $\Leftrightarrow$ $\Leftrightarrow$ surjectivity $\Leftrightarrow$ $\text{Rank}\, T = \dim V$.

$$\begin{array}{c} \| \\ 3. \end{array}$$

26. True.

$$\ker(T) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle/ T(a,b) = (0,0) \right\} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle/ (a,-b) = (0,0) \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle/ a = 0,\ b = 0 \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

27. False.

$$\operatorname{Im} T = \left\{ T\begin{pmatrix} a \\ b \end{pmatrix} \middle/ a, b \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \middle/ a, b \in \mathbb{R} \right\} =$$

$= \{$ All points in $\mathbb{R}^3$ having the y-coordinate zero $\}$ = the xz-plane

28. False.

If it would be surjective its image would be the whole $\mathbb{R}^3$, which does not happen.

29. True.

$$[T]_\alpha^\beta = \left(\left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_\beta \ \left[T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_\beta \right) = \left(\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]_\beta \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]_\beta \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

30. True.

$$\ker(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle/ T(a,b,c) = (0,0,0) \right\} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle/ (a,0,0) = (0,0,0) \right\} =$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle/ a = 0 \right\} = \left\{ \begin{pmatrix} 0 \\ b \\ c \end{pmatrix},\ b, c \in \mathbb{R} \right\}.$$