Exam 1 Solutions

1) a) $\operatorname{Im} T = P_2(\mathbb{R})$ (all polynomials in $P_2(\mathbb{R})$ represent the derivatives of some polynomials in $P_3(\mathbb{R})$).

$\dim \operatorname{Im} T = \dim P_2(\mathbb{R}) = 3$

b) $\dim \ker T + \underbrace{\dim \operatorname{Im} T}_{3} = \underbrace{\dim P_3(\mathbb{R})}_{4}$

$$\Downarrow$$

$\dim \ker T = 1$

c) $\ker T = \{ f \in P_3(\mathbb{R}) ; f' = 0 \} = \{$ The constant polynomials $\}$

2) a) $[T]_\alpha^\beta = \left([T(1)]_\beta \ [T(x)]_\beta \ [T(x^2)]_\beta \right) =$

$= \left([1]_\beta \ [2]_\beta \ [4]_\beta \right) = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$

b) $[T(2-3x+5x^2)]_\beta \overset{\text{def}}{\underset{T}{=}} [2 - 3\cdot 2 + 5 \cdot 2^2]_\beta = [16]_\beta = 16$

c) $[T(2-3x+5x^2)]_\beta = [T]_\alpha^\beta [2-3x+5x^2]_\alpha =$

$= \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = 1\cdot 2 + 2(-3) + 4\cdot 5 = 16$

(3) "$\Rightarrow$" Assume S independent, prove $T(S)$ independent.

Let $c_1, \dots, c_n \in \mathbb{R}$ and $T(\vec{v_1}), \dots, T(\vec{v_n}) \in T(S)$ s.t. $c_1 T(\vec{v_1}) + \dots + c_n T(\vec{v_n}) = \vec{0_W}$. To prove: $c_1 = \dots = c_n = 0$.

As T linear, $c_1 T(\vec{v_1}) + \dots + c_n T(\vec{v_n}) = T(c_1 \vec{v_1} + \dots + c_n \vec{v_n})$.

So $T(c_1 \vec{v_1} + \dots + c_n \vec{v_n}) = \vec{0_W}$. Injectivity gives $c_1 \vec{v_1} + \dots + c_n \vec{v_n} = \vec{0_V}$. But $\vec{v_1}, \dots, \vec{v_n} \in S$ who is independent, so $c_1 = \dots = c_n = 0$. $\Box$

"$\Leftarrow$" Assume $T(S)$ independent, prove S independent

Let $c_1, \dots, c_n \in \mathbb{R}$ and $\vec{v_1}, \dots, \vec{v_n} \in S$ s.t. $c_1 \vec{v_1} + \dots + c_n \vec{v_n} = \vec{0_V}$.

To prove: $c_1 = \dots = c_n = 0$.

From $c_1 \vec{v_1} + \dots + c_n \vec{v_n} = \vec{0_V}$ we get $T(c_1 \vec{v_1} + \dots + c_n \vec{v_n}) = \vec{0_W}$.

As T linear, $T(c_1 \vec{v_1} + \dots + c_n \vec{v_n}) = c_1 T(\vec{v_1}) + \dots + c_n T(\vec{v_n})$.

So $c_1 T(\vec{v_1}) + \dots + c_n T(\vec{v_n}) = \vec{0_W}$. But $T(\vec{v_1}), \dots, T(\vec{v_n}) \in T(S)$ who is independent, so $c_1 = \dots = c_n = 0$. $\Box$.

(4) a) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) =$

linear.

$$= x \underbrace{T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\substack{\text{I'll denote} \\ a \in \mathbb{R}}} + y \underbrace{T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\substack{\text{I'll denote} \\ b \in \mathbb{R}}} + z \underbrace{T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\substack{\text{I'll denote} \\ c \in \mathbb{R}}} = ax + by + cz.$$ So there $\exists$

(3)

$a, b, c \in \mathbb{R}$ s.t. $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$.

b) If T is the zero transformation, $\dim \operatorname{Im} T = \dim \{0_{\mathbb{R}}\} = 0$.
If T is $\neq$ the zero transformation, there $\exists$ at least a nonzero $\vec{v}$ in $\operatorname{Im} T$, so $\dim \operatorname{Im} T \geq 1$. But also $\dim T \leq 1$ (as $\dim \mathbb{R} = 1$). So $\dim T = 1$.
Conclusion: possible dimensions of $\operatorname{Im} T$ are 0 and 1.

c) $\dim \ker T + \dim \operatorname{Im} T = 3$, so when $\dim \operatorname{Im} T = 0$, $\dim \ker T$ will be 3, when $\dim \operatorname{Im} T = 1$, $\dim \ker T$ will be 2.

d) No, as $\dim \mathbb{R}^3 > \dim \mathbb{R}$ (ex. 17 /2.1)

(5) For such a T we have:
$\operatorname{Im} T^2 = T(T(\mathbb{R}^2)) = T(\operatorname{Im} T) = T(\ker T) = \{0_{\mathbb{R}^2}\}$
So T^2 = The 0 transformation. Hence $[T^2]_\alpha = 0$ (the zero matrix)
α: canonical basis

But $[T^2]_\alpha = [T]_\alpha [T]_\alpha$, so $[T]_\alpha [T]_\alpha = 0$. Hence $[T]_\alpha$ can be for instance $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$, etc.

If $[T]_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, T is the transformation $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$.
If $[T]_\alpha = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, T is the transformation $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ a-b \end{pmatrix}$

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If $[T]_\alpha = \begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}$, T is the transformation $T\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}b-a\\b-a\end{pmatrix}$.

All these three transformations satisfy $\ker T = \operatorname{Im} T$.

For the first one $\ker T = \operatorname{Im} T =$ x-axis

-//- second -//- $\ker T = \operatorname{Im} T =$ The line $x = y$

-//- third one $\ker T = \operatorname{Im} T =$ The line $x = y$.

Part 2 :

6 $[T(\vec{v})]_\beta = [T]_\alpha^\beta [\vec{v}]_\alpha$, so $\vec{v}$ is the basis α, not β.

False

7 $\underset{\dim m}{L(V,}\ \underset{\dim n}{W)} \simeq M_{n\times m}(\mathbb{R})$, so the assertion

at 7 is False.

:)

8 False. $\{1, x, x^2, x^3\}$ is a basis for $P_3(\mathbb{R})$,

but $\{1', x', (x^2)', (x^3)'\} = \{0, 1, 2x, 3x^2\}$ which

is not a basis for $P_2(\mathbb{R})$ (contains 0, so is dependent)

(9) True

As $V = W_1 \oplus W_2$, we have $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\vec{0}_v\}$. So any $\vec{v} \in V$ can be written as $\vec{v} = \vec{v_1} + \vec{v_2}$, with $\vec{v_1} \in W_1$ and $\vec{v_2} \in W_2$.

If there would exist also $\vec{v_1}' \in W_1$ and $\vec{v_2}' \in W_2$ s.t. $\vec{v} = \vec{v_1}' + \vec{v_2}'$, we would have:

$$\vec{v_1} + \vec{v_2} = \vec{v_1}' + \vec{v_2}'$$

hence

$$\underbrace{\vec{v_1} - \vec{v_1}'}_{\in W_1} = \underbrace{\vec{v_2}' - \vec{v_2}}_{\in W_2}$$

they havo only the $\vec{0_v}$ in commun.

So $\vec{v_1} - \vec{v_1}' = \vec{v_2}' - \vec{v_2} = \vec{0_v}$, so $\vec{v_1}' = \vec{v_1}$, $\vec{v_2}' = \vec{v_2}$, thus the representation of $\vec{v}$ as $\underset{\in W_1}{\vec{v_1}} + \underset{\in W_2}{\vec{v_2}}$ is unique.

(10) True $\operatorname{Dim} \mathcal{L}(\underset{\dim 2}{V}, \underset{\dim 2}{W}) = \operatorname{Dim}(M_{2\times 2}(\mathbb{R})) = 4$. (1)

$\operatorname{Dim} \mathbb{R}^4 = 4$ (2)

From (1) and (2) $\Rightarrow \mathcal{L}(V, W) \simeq \mathbb{R}^4$.

11 True, because if we would have $T(\vec{v}) = U(\vec{v})$ for all $\vec{v} \in \beta$ we would have $T = U$ according to the corollary of th. 2.6.

12 True Th. 2.6 $\Rightarrow \exists$ a linear transform. $T: V \to W$ s.t. $T(\vec{v_1}) = \vec{w_1}, \ldots, T(\vec{v_n}) = \vec{w_n}$. This means $T(\{\vec{v_1}, \ldots, \vec{v_n}\}) = \{\vec{w_1}, \ldots, \vec{w_n}\}$ which is subset of $\{\vec{w_1}, \ldots, \vec{w_m}\}$ $(m > n)$.

13 True. If $[T]_\alpha^\beta$ is invertible, the T is invertible, so bijective, so injective, so has $\ker T = \{\vec{0}\}$.

14 True $\underset{P(R)}{T(a_0 + a_1 x + \cdots + a_n x^n)} \stackrel{\text{def}}{=} \underset{P_2(R)}{a_0 + a_1 x + a_2 x^2}$.

This T is nonzero and is linear.

15 False, because $[TU]_\alpha \stackrel{\text{def}}{=} [TU]_\alpha^\alpha = [T]_\beta^\alpha [U]_\alpha^\beta$.

Part 3 :

16 To prove: $\operatorname{Im} T = \operatorname{span} T(\beta)$, where $\beta = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis in V.

"$\subseteq$" Let $\vec{w} \in \operatorname{Im} T$. By def. of $\operatorname{Im} T$, there $\exists\ \vec{v} \in V$ s.t. $\vec{w} = T(\vec{v})$. But β is basis in V, so $\vec{v} = c_1\vec{v_1} + \ldots + c_n\vec{v_n}$

Hence $\vec{w} = T(c_1\vec{v_1} + \ldots + c_n\vec{v_n}) \overset{\text{linearity}}{=} c_1 T(\vec{v_1}) + \ldots + c_n T(\vec{v_n})$ $\in \operatorname{span}\{T(\vec{v_1}), \ldots, T(\vec{v_n})\} = \operatorname{span} T(\beta)$. $\square$

"$\supseteq$" As $\operatorname{Im} T$ subspace in W, sufficient to prove that $\operatorname{Im} T \supseteq T(\beta)$ (because if a subspace includes a set S he will also include the span S).

So let $T(\vec{v_i}) \in T(\beta)$. By def. of $\operatorname{Im} T$ we have $T(\vec{v_i}) \in \operatorname{Im} T$. Thus $T(\beta) \subseteq \operatorname{Im} T$.

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