

①

Exam 1 Solutions

① a) $\text{Im } T = P_2(\mathbb{R})$ (all polynomials in $P_2(\mathbb{R})$ represent the derivatives of some polynomials in $P_3(\mathbb{R})$).

$$\dim \text{Im } T = \dim P_2(\mathbb{R}) = 3$$

$$b) \dim \ker T + \dim \text{Im } T = \dim P_3(\mathbb{R})$$

$$\begin{array}{ccc} & 4 & \\ & 3 & \\ & & 4 \end{array}$$

$$\downarrow \\ \dim \ker T = 1$$

$$c) \ker T = \{f \in P_3(\mathbb{R}); f' = 0\} = \{\text{The constant polynomials}\}$$

$$\textcircled{2} \quad a) [T]_{\beta}^{\beta} = \left([T(1)]_{\beta} \quad [T(x)]_{\beta} \quad [T(x^2)]_{\beta} \right) = \\ = \left([1]_{\beta} \quad [2]_{\beta} \quad [4]_{\beta} \right) = (1 \quad 2 \quad 4)$$

$$b) [T(2-3x+5x^2)]_{\beta} \stackrel{\text{def}}{=} [T]_{\beta}^{\beta} [2-3x+5x^2]_{\beta} = [16]_{\beta} = 16$$

$$c) [T(2-3x+5x^2)]_{\beta} = [T]_{\beta}^{\beta} [2-3x+5x^2]_{\beta} = \\ = (1 \quad 2 \quad 4) \cdot \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = 1 \cdot 2 + 2(-3) + 4 \cdot 5 = 16$$

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(2)

(3) " \Rightarrow " Assume S independent, prove $T(S)$ independent.

Let $c_1, \dots, c_n \in \mathbb{R}$ and $T(\vec{v}_1), \dots, T(\vec{v}_n) \in T(S)$ s.t.

$c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n) = \vec{0}_w$. To prove: $c_1 = \dots = c_n = 0$.

As T linear, $c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n) = T(c_1 \vec{v}_1 + \dots + c_n \vec{v}_n)$.

So $T(c_1 \vec{v}_1 + \dots + c_n \vec{v}_n) = \vec{0}_w$. Injectivity gives

$c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{0}_v$. But $\vec{v}_1, \dots, \vec{v}_n \in S$ who is

independent, so $c_1 = \dots = c_n = 0$. \square

" \Leftarrow " Assume $T(S)$ independent, prove S independent

Let $c_1, \dots, c_n \in \mathbb{R}$ and $\vec{v}_1, \dots, \vec{v}_n \in S$ s.t. $c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{0}_v$.

To prove: $c_1 = \dots = c_n = 0$.

From $c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{0}_v$ we get $T(c_1 \vec{v}_1 + \dots + c_n \vec{v}_n) = \vec{0}_w$.

As T linear, $T(c_1 \vec{v}_1 + \dots + c_n \vec{v}_n) = c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n)$.

So $c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n) = \vec{0}_w$. But $T(\vec{v}_1), \dots, T(\vec{v}_n)$

$\in T(S)$ who is independent, so $c_1 = \dots = c_n = 0$. \square .

(4) a) $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = T\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) =$

linear:

$$= x T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + y T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + z T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = ax + by + cz. \text{ So there } \exists$$

$\underbrace{\hspace{1.5cm}}_{\text{denote } a \in \mathbb{R}} \quad \underbrace{\hspace{1.5cm}}_{\text{denote } b \in \mathbb{R}} \quad \underbrace{\hspace{1.5cm}}_{\text{denote } c \in \mathbb{R}}$

a, b, c $\in \mathbb{R}$ s.t. $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$.

b) If T is the zero transformation, $\dim \text{Im } T = \dim \{0_{\mathbb{R}}\} = 0$.

If T is \neq the zero transformation, there is at least a nonzero \vec{v} in $\text{Im } T$, so $\dim \text{Im } T \geq 1$. But also $\dim T \leq 1$ (as $\dim \mathbb{R} = 1$). So $\dim T = 1$.

Conclusion: possible dimensions of $\text{Im } T$ are 0 and 1.

c) $\dim \text{Ker } T + \dim \text{Im } T = 3$, so when $\dim \text{Im } T = 0$, $\dim \text{Ker } T$ will be 3, when $\dim \text{Im } T = 1$, $\dim \text{Ker } T$ will be 2.

d) No, as $\dim \mathbb{R}^3 > \dim \mathbb{R}$ (ex. 17 / 2.1)

5) For such a T we have:

$$\text{Im } T^2 = T(T(\mathbb{R}^2)) = T(\text{Im } T) = T(\text{Ker } T) = \{0_{\mathbb{R}^2}\}$$

So $T^2 =$ The 0 transformation. Hence $[T^2]_{\alpha} = 0$ (the zero matrix)
canonical basis

But $[T^2]_{\alpha} = [T]_{\alpha} [T]_{\alpha}$, so $[T]_{\alpha} [T]_{\alpha} = 0$. Hence $[T]_{\alpha}$ can be for instance $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$, etc.

If $[T]_{\alpha} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, T is the transformation $T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$.

If $[T]_{\alpha} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, T is the transformation $T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ a-b \end{pmatrix}$.

(4)

If $[T]_{\alpha} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$, T is the transformation $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b-a \\ b-a \end{pmatrix}$.

All these three transformations satisfy $\ker T = \text{Im } T$.

For the first one $\ker T = \text{Im } T = x$ -axis

-||- second -||- $\ker T = \text{Im } T =$ The line $x=y$

-||- third one $\ker T = \text{Im } T =$ The line $x=y$.

Part 2 :

(6) $[T(\vec{v})]_{\beta} = [T]_{\alpha}^{\beta} [\vec{v}]_{\alpha}$, so \vec{v} is the basis α , not β .

False

(7) $L(\underset{\dim m}{V}, \underset{\dim n}{W}) \cong \mathcal{M}_{n \times m}(\mathbb{R})$, so the assertion

at (7) is False.

(8) False. $\{1, x, x^2, x^3\}$ is a basis for $P_3(\mathbb{R})$,
but $\{1', x', (x^2)', (x^3)'\} = \{0, 1, 2x, 3x^2\}$ which
is not a basis for $P_2(\mathbb{R})$ (contains 0, so is dependent)

(5)

(9) True

As $V = W_1 \oplus W_2$, we have $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\vec{0}_V\}$. So any $\vec{v} \in V$ can be written as $\vec{v} = \vec{v}_1 + \vec{v}_2$, with $\vec{v}_1 \in W_1$ and $\vec{v}_2 \in W_2$. If there would exist also $\vec{v}_1' \in W_1$ and $\vec{v}_2' \in W_2$ s. t. $\vec{v} = \vec{v}_1' + \vec{v}_2'$, we would have:

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_1' + \vec{v}_2'$$

hence

$$\underbrace{\vec{v}_1 - \vec{v}_1'}_{\in W_1} = \underbrace{\vec{v}_2' - \vec{v}_2}_{\in W_2}$$



they have only the $\vec{0}_V$ in common.

$$\text{So } \vec{v}_1 - \vec{v}_1' = \vec{v}_2' - \vec{v}_2 = \vec{0}_V, \text{ so } \vec{v}_1' = \vec{v}_1, \vec{v}_2' = \vec{v}_2,$$

thus the representation of \vec{v} as $\vec{v}_1 + \vec{v}_2$ is unique.

(10) True

$$\dim L(V, W) = \dim (M_{2 \times 2}(\mathbb{R})) = 4.$$

$$\dim \mathbb{R}^4 = 4 \quad (2)$$

From (1) and (2) $\Rightarrow L(V, W) \cong \mathbb{R}^4$.

6

(11) (True), because if we would have $T(\vec{v}^i) = U(\vec{v}^i)$ for all $\vec{v}^i \in \beta$ we would have $T = U$ according to the corollary of th. 2.6.

(12) (True) Th. 2.6 $\Rightarrow \exists$ a linear transform $T: V \rightarrow W$ s.t. $T(\vec{v}_1) = \vec{w}_1, \dots, T(\vec{v}_n) = \vec{w}_n$. This means $T(\{\vec{v}_1, \dots, \vec{v}_n\}) = \{\vec{w}_1, \dots, \vec{w}_n\}$ which is subset of $\{\vec{w}_1, \dots, \vec{w}_m\}$ ($m > n$).

(13) (True), If $[T]_{\alpha}^{\beta}$ is invertible, the T is invertible, so bijective, so injective, so has $\ker T = \{\vec{0}\}$.

(14) (True) $T(a_0 + a_1x + \dots + a_nx^n) \stackrel{\text{def}}{=} a_0 + a_1x + a_2x^2$.

$\mathbb{P}(\mathbb{R})$ $\mathbb{P}_2(\mathbb{R})$

This T is nonzero and is linear.

(15) (False), because $[TU]_{\alpha} \stackrel{\text{def}}{=} [TU]_{\alpha}^{\alpha} = [T]_{\beta}^{\alpha} [U]_{\alpha}^{\beta}$.

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(7)

Part 3:

(16) To prove: $\text{Im } T = \text{span } T(\beta)$, where

$\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis in V .

" \subseteq " Let $\vec{w} \in \text{Im } T$. By def. of $\text{Im } T$, there $\exists \vec{v} \in V$

s.t. $\vec{w} = T(\vec{v})$. But β is basis in V , so $\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$

Hence $\vec{w} = T(c_1 \vec{v}_1 + \dots + c_n \vec{v}_n) \stackrel{\text{linearity}}{=} c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n)$

$\in \text{span } \{T(\vec{v}_1), \dots, T(\vec{v}_n)\} = \text{span } T(\beta)$. \square

" \supseteq " As $\text{Im } T$ subspace in W , sufficient to prove that $\text{Im } T \supseteq T(\beta)$ (because if a subspace includes a set S he will also include the span S).

So let $T(\vec{v}_i) \in T(\beta)$. By def. of $\text{Im } T$ we have $T(\vec{v}_i) \in \text{Im } T$. Thus $T(\beta) \subseteq \text{Im } T$.

\square .

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