IE 230 Seat # ________ Name ___ < KEY > ___

Closed book and notes. 120 minutes.

Cover page, five pages of exam.
No calculator. No need to simplify answers.

(2 points) I have, or will, complete a course evaluation.

____
...sign here...

Score ___ < ??? / 102 > ___

Closed book and notes. 120 minutes.

Throughout this exam, consider an experiment that chooses a random sample $X_1, X_2, \ldots, X_n$ from a population with cdf F, mean $\mathrm{E}(X) = \mu$, variance $\mathrm{V}(X) = \sigma^2$ and median $x_{0.5} = F^{-1}(0.5)$. Let $\bar{X}$ denote the sample mean, $S^2 = [\sum_{i=1}^{n} (X_i - \bar{X})^2] / (n-1)$ denote the sample variance, and $X_{(i)}$ the ith order statistic.

1. True or false. (2 points each)

(a) T ← F A random sample contains independent observations.

(b) T ← F A random sample contains identically distributed observations.

(c) T ← F The abbreviation "iid" means independent and identically distributed.

(d) T F ← A point estimate is a random variable; a point estimator is a number.

(e) T F ← Independent observations are always identically distributed.

(f) T F ← Identically distributed observations are always independent.

(g) T F ← Independent observations can be mutually exclusive.

(h) T ← F A linear combination of random variables is a random variable.

(i) T F ← A population is used to infer conclusions about a sample.

(j) T ← F For a distribution with a symmetric pdf, the median and mean are equal.

2. (3 points each) Fill in the blanks with course jargon (a word or phrase).

(a) The standard deviation of a point estimator is its ___ < standard error > ___ .

(b) A random variable computed from a sample is a ___ < statistic > ___ .

(c) A statistic used to estimate a population parameter is a ___ < point estimator > ___ .

(d) A sample is selected from a ___ < population > ___ .

3. Result: If $X_1, X_2, \ldots, X_n$ is a random sample from a population with cdf F_X and mean μ_X and variance σ_X^2, then the sample variance, $S_X^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1)$, is an unbiased estimator of σ_X^2.

For each line of the proof, choose from these reasons. (2 points each)

(i) Axiom 3: If events A and B are mutually exclusive, then $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.
(ii) Independence
(iii) Algebra (no probability required)
(iv Definition of expected value
(v) Definition of sample mean
(vi) $\mathrm{V}(X) = \mathrm{E}(X^2) - \mu_X^2$
(vii) For a random sample, $\mathrm{V}(\bar{X}) = \sigma_X^2 / n$
(viii) Same event
(ix) Random sample from F_X
(x) Expected value of a linear combination
(xi) Definition of mean squared error
(xii) Definition of sample variance

$$\mathrm{E}(S_X^2) = \mathrm{E}\left[\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}\right]$$ (a) ___ < xii > ___

$$= \mathrm{E}\left[\frac{\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X}^2)}{n-1}\right]$$ (b) ___ < iii > ___

$$= \mathrm{E}\left[\frac{(\sum_{i=1}^{n} X_i^2) - 2\bar{X}(\sum_{i=1}^{n} X_i) + n\bar{X}^2}{n-1}\right]$$ (c) ___ < iii > ___

$$= \mathrm{E}\left[\frac{(\sum_{i=1}^{n} X_i^2) - 2\bar{X}(n\bar{X}) + n\bar{X}^2}{n-1}\right]$$ (d) ___ < v > ___

$$= \mathrm{E}\left[\frac{(\sum_{i=1}^{n} X_i^2) - n\bar{X}^2}{n-1}\right]$$ (e) ___ < iii > ___

$$= \frac{\sum_{i=1}^{n} \mathrm{E}(X_i^2) - n\,\mathrm{E}[\bar{X}^2]}{n-1}$$ (f) ___ < x > ___

$$= \frac{\sum_{i=1}^{n} (\sigma_X^2 + \mu_X^2) - n\,(\sigma_{\bar{X}}^2 + \mu_X^2)}{n-1}$$ (g) ___ < vi > ___

$$= \frac{n\,(\sigma_X^2 + \mu_X^2) - n\,(\sigma_{\bar{X}}^2 + \mu_X^2)}{n-1}$$ (h) ___ < iii > ___

$$= \frac{n\,(\sigma_X^2 + \mu_X^2) - n(\sigma_X^2/n + \mu_X^2)}{n-1}$$ (i) ___ < vii > ___

$$= \sigma_X^2$$ (j) ___ < iii > ___

4. (2 points each) Choose one answer.

(a) S_X^2	constant	random variable ←	event	meaningless
(b) $\mathrm{E}[S_X^2]$	constant ←	random variable	event	meaningless
(c) $n-1$	constant ←	random variable	event	meaningless
(d) $\hat{\sigma}_{\bar{X}}^2$	constant	random variable ←	event	meaningless

5. (From Montgomery and Runger, fourth edition, 7–14) Suppose that we have a random sample of size 10 from a population denoted by X, and that $\mathrm{E}(X) = \mu$ and $\mathrm{V}(X) = \sigma^2$. Let $\bar{X}_1 = (X_1 + X_2 + X_3 + X_4 + X_5)/5$ and $\bar{X}_2 = (X_1 + X_3 + X_5 + X_7 + X_9)/5$.

(a) (2 points) T ← F $\mathrm{E}(\bar{X}_1) = \mathrm{E}(\bar{X}_2)$.

(b) (2 points) T ← F $\mathrm{Var}(\bar{X}_1) = \mathrm{Var}(\bar{X}_2)$.

(c) (2 points) Suggest a better point estimator for μ, say $\bar{X}_3$.

$\bar{X}_3 = 10^{-1} \sum_{i=1}^{10} X_i$ ←

6. (2 points each) We discussed the analogy of point estimators being like target shooting. Error in target shooting is caused by either aiming the wrong direction or by not holding the gun steady. Error in point estimation is caused by bias and standard error.

(a) T ← F Mean squared error is squared bias plus squared standard error.

(b) T F ← Standard error is analogous to aiming in the wrong direction.

7. A computer software package was used to calculate some numerical summaries of a random sample of data from a population with mean μ and variance σ^2. The results are in the following table.

Variable	n	mean	ste mean	stdev	variance
X	20	20.184	?	1.816	?

(a) (5 points) (first missing value) What is the value of the estimated standard error of the sample mean?

$1.816 / \sqrt{20} \leftarrow$

(b) (5 points) (second missing value) What is the value of the sample variance?

$1.816^2 \leftarrow$

(c) (5 points) What is the value of the point estimate of μ?

$20.184 \leftarrow$

8. Consider the normal distribution with mean $\mu = 12$ and standard deviation $\sigma = 2$. A sample of size $n = 3$ is taken, with observations $x_1 = 11.5$, $x_2 = 10.5$, $x_3 = 13.0$.

(a) (5 points) With substantial care, sketch the density function of this normal distribution. Label and scale both axes.

Sketch the usual bell curve, centered at 12, points of inflection at 10 and 14, disappearing at about 6 and 18.

Label the horizontal axis with x; the vertical axis with $f_X(x)$.

Scale the horizontal axis with at least two numbers.

Scale the vertical axis with at least two numbers, probably zero and the height of the mode, $.4/\sigma = 0.2$.

(b) (5 points) In your sketch, indicate the likelihood of x_1, of x_2, and of x_3.

Plot the three values $x_1 = 11.5$, $x_2 = 10.5$, $x_3 = 13.0$ on the horizontal axis.

The three likelihoods are the three values $f_X(11.5)$, $f_X(10.5)$, and $f_X(13.0)$.

(c) (5 points) Determine the method-of-moments estimator of μ.

Set $\mathrm{E}(X) = \bar{X}$.

Therefore, $\hat{\mu} = \dfrac{11.5 + 10.5 + 13.0}{3} \leftarrow$
