IE 230 Seat # ________ Name ___ < KEY > ___

Closed book and notes. 60 minutes.

Cover page and four pages of exam.
Pages 8 and 12 of the Concise Notes.
No calculator. No need to simplify answers.

This test is cumulative, with emphasis on Section 4.8
through Section 5.5 of Montgomery and Runger, fourth edition.

Remember: A statement is true only if it is always true.

One point: On the cover page, circle your family name.

One point: On every page, write your name.

The random vector $(X_1, X_2, \ldots, X_k)$ has a *multinomial distribution* with joint pmf

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = \frac{n!}{x_1!\, x_2! \cdots x_k!}\, p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

when each x_i is a nonnegative integer and $x_1 + x_2 + \cdots + x_k = n$; zero elsewhere.

The linear combination $Y = c_0 + c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$ has mean and variance

$$E(Y) = c_0 + \sum_{i=1}^{n} E(c_i X_i) = c_0 + \sum_{i=1}^{n} c_i\, E(X_i)$$

and

$$V(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \operatorname{cov}(c_i X_i, c_j X_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i\, c_j \operatorname{cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} c_i^2\, V(X_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_i\, c_j \operatorname{cov}(X_i, X_j)\ .$$

$\operatorname{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$

$\operatorname{Corr}(X, Y) = \operatorname{Cov}(X, Y) / (\sigma_X\, \sigma_Y)$

If (X, Y) is bivariate normal, then X and Y are normal. In addition, the conditional distribution of X given that $Y = y$ is normal, with mean $\mu_X + \rho_{X,Y}\, \sigma_X\, [(y - \mu_Y)/\sigma_Y]$ and variance $(1 - \rho_{X,Y}^2)\, \sigma_X^2$.

Score ______________________________

Closed book and notes. 60 minutes.

1. (3 points each) True or false.

Consider the notes from the cover page.

(a) T F ← $\mathrm{E}(X - Y) = \mu_X + \mu_Y$.

(b) T ← F Corr$(X, Y) \geq 0$ implies that Cov$(X, Y) \geq 0$.

(c) T F ← If (X, Y) is bivariate normal, then $\rho_{X,Y} = 0$.

(d) T F ← If X_1 and X_2 are independent and exponentially distributed, then $X_1 + X_2$ has an exponential distribution.

(e) T ← F If (X, Y) is bivariate normal, then $4.5(X + Y)$ is normally distributed.

(f) T F ← If v and w are real numbers, then $f_{X,Y}(v, w) = f_X(v)\, f_Y(w)$.

(g) T ← F If X and Y are independent, then $\rho_{X,Y} = 0$.

(h) T ← F The time to recharge a battery is never normally distributed.

2. (3 points each) Consider the notation from the cover page. For each of the following, indicate whether the expression is a constant, an event, a random variable, or undefined. (A constant has the same numerical value for every replication of the experiment.)

(a) $\rho_{X,Y}$ constant ← event random variable undefined

(b) $\mu_X(3)$ constant event random variable undefined ←

(c) $\mathrm{P}(X)$ constant event random variable undefined ←

(d) $\mathrm{E}(X = 3)$ constant event random variable undefined ←

3. (Problem 4–91, Montgomery and Runger, fourth edition) Phone calls to a corporate office follow a Poisson process with rate six per hour. The times between calls, then, are independent and exponential.

(a) (4 points) Determine the mean time between calls.

The Poisson rate is $\lambda = 6$ calls per hour.
The mean time between calls is $1/\lambda = 1/6$ hour = 10 minutes ←

(b) (5 points) Suppose that it is now 11:30 am. Determine the most-likely (clock) time of the next phone call.

The time until the next call is exponential with rate λ.
The mode of the exponential pdf is at time zero.
Therefore, the most-likely time is 11:30am. ←

(c) (5 points) Suppose that it is now 11:30 am. Determine the expected (clock) time of the phone call after next.

The time until the call after next is Erlang, with $r = 2$ and rate $\lambda = 6$ calls per hour.
From Part (a), the Erlang mean is $r/\lambda = 20$ minutes.
Therefore, the expected time of the next call is 11:50am. ←

(d) (5 points) Suppose that it is now 11:30 am. The previous phone call occurred at 11:22am. Determine the expected (clock) time of the next phone call.

The time until the next call is exponential with rate $\lambda = 6$ calls per hour.
Exponential distributions are memoryless, so the time of the previous call is irrelevant.
The expected time until the next phone call is 10 minutes.
Therefore, the expected time of the next phone call is 11:40 am. ←

---

4. (Problem 5–35, Montgomery and Runger, fourth edition) Consider the probability mass function $f_{X,Y}(x,y) = c\,(x+y)$ for $x = 1, 2, 3$ and $y = 1, 2, 3$ and zero elsewhere.

(a) (5 points) Determine the value of c.

The random variables X and Y are discrete.
Therefore, $\sum_{x=1}^{3}\sum_{y=1}^{3} f_{X,Y}(x,y) = 1$.
That is, $\sum_{x=1}^{3}\sum_{y=1}^{3} c\,(x+y) = 1$.
Solving for c yields $c = [2+3+4+3+4+5+4+5+6]^{-1} = 1/36 \leftarrow$

(b) (5 points) Determine the value of $\mathrm{P}(X \le 0,\, 1 \le Y \le 3)$.

X takes only the values 1, 2, and 3. Therefore,

$$\mathrm{P}(X \le 0,\, 1 \le Y \le 3) = 0 \leftarrow$$

(c) (5 points) Determine the value of $\mathrm{P}(X = 1 \mid Y = 3)$.

$$\begin{aligned}
\mathrm{P}(X = 1 \mid Y = 3) &= \mathrm{P}(X = 1, Y = 3) \,/\, \mathrm{P}(Y = 3) \\
&= \mathrm{P}(X=1, Y=3) / [\mathrm{P}(X=1, Y=3) + \mathrm{P}(X=2, Y=3) + \mathrm{P}(X=3, Y=3)] \\
&= c\,(1+3) / [c\,(1+3) + c\,(2+3) + c\,(3+3)] \\
&= 4/[4+5+6] \\
&= 4/15 \leftarrow
\end{aligned}$$

---

5. (Problem 5–64, Montgomery and Runger, fourth edition) Assume that the weights of persons are independent and normally distributed, each with mean 160 pounds and standard deviation 30 pounds. Suppose that 25 persons squeeze into an elevator that is designed to hold 4300 pounds. Let X_i denote the weight of person i, for $i = 1,2,\ldots,25$.

(a) (4 points) Describe the experiment.

Squeeze 25 persons into the elevator. $\leftarrow$

(b) (4 points) Using the given notation, write the total weight of the twenty-five persons.

$Y = \sum_{i=1}^{25} X_i = X_1 + X_2 + \cdots + X_{25} \leftarrow$

(c) (5 points) Determine the expected total weight.

$\mathrm{E}(Y) = (25)\,\mu_X = 4000$ pounds $\leftarrow$

(d) (5 points) Determine the total-weight's standard deviation.

$\mathrm{Var}(Y) = (25)\sigma_X^2 = (25)(30^2)$ pounds squared.

Therefore, $\mathrm{std}(Y) = (5)(30) = 150$ pounds $\leftarrow$

(e) (5 points) Determine the probability that the total weight is greater than the elevator's capacity.

$\mathrm{P}(Y > 4300) = \mathrm{P}(Z > 2) = 1 - \Phi(2) \approx 1 - 0.975 = 0.025 \leftarrow$

(Here $Z = (Y - \mu_Y)/\sigma_Y$ and Φ is the standard-normal cdf.)

(f) (5 points) Suppose that you are one of the 25 persons. You know that your weight is 180 pounds. Conditional on that knowledge, what is your answer to Part (c)?

Without loss of generality, assume that you are the 25th person. Then $X_{25} = 180$.
The total weight is still $Y = X_1 + X_2 + \cdots + X_{24} + X_{25}$.
The conditional mean total weight is $\mathrm{E}(Y \mid X_{25} = 180) = (24)(160) + 180 = 4020$ pounds $\leftarrow$

Discrete Distributions: Summary Table

random variable	distribution name	range	probability mass function	expected value	variance
X	general	$x_1, x_2, \ldots, x_n$	$P(X = x)$ $= f(x)$ $= f_X(x)$	$\sum_{i=1}^{n} x_i f(x_i)$ $= \mu = \mu_X$ $= E(X)$	$\sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	discrete uniform	$x_1, x_2, \ldots, x_n$	$1/n$	$\sum_{i=1}^{n} x_i / n$	$[\sum_{i=1}^{n} x_i^2 / n] - \mu^2$
X	equal-space uniform	$x = a, a+c, \ldots, b$ where	$1/n$ $n = (b-a+c)/c$	$\frac{a+b}{2}$	$\frac{c^2(n^2-1)}{12}$
"# successes in 1 Bernoulli trial"	indicator variable	$x = 0, 1$	$p^x(1-p)^{1-x}$	p where	$p(1-p)$ $p = P(\text{"success"})$
"# successes in n Bernoulli trials"	binomial	$x = 0, 1, \ldots, n$	$C_x^n p^x (1-p)^{n-x}$	np where	$np(1-p)$ $p = P(\text{"success"})$
"# successes in a sample of size n from a population of size N containing K successes"	hyper-geometric (sampling without replacement)	$x =$ $(n-(N-K))^+$, ..., $\min\{K, n\}$ and integer	$C_x^K C_{n-x}^{N-K} / C_n^N$	np where	$np(1-p)\frac{(N-n)}{(N-1)}$ $p = K/N$
"# Bernoulli trials until 1st success"	geometric	$x = 1, 2, \ldots$	$p(1-p)^{x-1}$	$1/p$ where	$(1-p)/p^2$ $p = P(\text{"success"})$
"# Bernoulli trials until rth success"	negative binomial	$x = r, r+1, \ldots$	$C_{r-1}^{x-1} p^r (1-p)^{x-r}$	r/p where	$r(1-p)/p^2$ $p = P(\text{"success"})$
"# of counts in time t from a Poisson process with rate λ"	Poisson	$x = 0, 1, \ldots$	$e^{-\mu} \mu^x / x!$	μ where	μ $\mu = \lambda t$

Result. For $x = 1, 2, \ldots$, the geometric cdf is $F_X(x) = 1 - (1-p)^x$.

Result. The geometric distribution is the only discrete memoryless distribution. That is, $P(X > x + c \mid X > x) = P(X > c)$.

Result. The binomial distribution with $p = K/N$ is a good approximation to the hypergeometric distribution when n is small compared to N.

Continuous Distributions: Summary Table

random variable	distribution name	range	cumulative distrib. func.	probability density func.	expected value	variance
X	general	$(-\infty, \infty)$	$P(X \le x)$ $= F(x)$ $= F_X(x)$	$\left.\frac{dF(y)}{dy}\right\|_{y=x}$ $= f(x)$ $= f_X(x)$	$\int_{-\infty}^{\infty} x f(x) dx$ $= \mu = \mu_X$ $= E(X)$	$\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	continuous uniform	$[a, b]$	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
X	triangular	$[a, b]$	$(x-a)f(x)/2$ if $x \le m$, else $1-(b-x)f(x)/2$	$\frac{2(x-d)}{(b-a)(m-d)}$ ($d = a$ if $x \le m$, else $d = b$)	$\frac{a+m+b}{3}$	$\frac{(b-a)^2-(m-a)(b-m)}{18}$
sum of random variables	normal (or Gaussian)	$(-\infty, \infty)$	Table III	$\frac{e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}$	μ	σ^2
time to Poisson count 1	exponential	$[0, \infty)$	$1 - e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
time to Poisson count r	Erlang	$[0, \infty)$	$\sum_{k=r}^{\infty} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$	r/λ	r/λ^2
lifetime	gamma	$[0, \infty)$	numerical	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$	r/λ	r/λ^2
lifetime	Weibull	$[0, \infty)$	$1 - e^{-(x/\delta)^\beta}$	$\frac{\beta x^{\beta-1} e^{-(x/\delta)^\beta}}{\delta^\beta}$	$\delta\Gamma(1+\frac{1}{\beta})$	$\delta^2\Gamma(1+\frac{2}{\beta})-\mu^2$

Definition. For any $r > 0$, the *gamma function* is $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx$.

Result. $\Gamma(r) = (r-1)\Gamma(r-1)$. In particular, if r is a positive integer, then $\Gamma(r) = (r-1)!$.

Result. The exponential distribution is the only continuous memoryless distribution.
That is, $P(X > x + c \mid X > x) = P(X > c)$.

Definition. A *lifetime* distribution is continuous with range $[0, \infty)$.

Modeling lifetimes. Some useful lifetime distributions are the exponential, Erlang, gamma, and Weibull.