5pts. for your Name

Closed book and notes. 60 minutes.

20 pts. total

1. True or false. (for each, 2 points if correct, 1 point if left blank.)

2 pts. each

(a) T F The binomial distribution is a special case of the multinomial distribution.

(b) T F The normal density function has only one shape.

(c) T F The exponential distribution is a special case of the normal distribution.

(d) T F For a fixed probability of success, the normal approximation to the binomial distribution is best when the number of trials is large.

(e) T F If Poisson occurrences have rate λ, then the average time between occurrences is also λ.

(f) T F $\Gamma(4) = 4!$

(g) T F When using the normal distribution to approximate the binomial distribution, an example of the continuity correction is $P(3 \le X \le 6) = P(2.5 \le X \le 6.5)$.

(h) T F Knowing the marginal distributions of X and Y is sufficient information to compute the joint distribution of (X, Y).

(i) T F The exponential distribution is the special case of the gamma distribution with shape parameter $r = 1$.

(j) T F The exponential distribution is the special case of the Weibull distribution with shape parameter $\beta = 1$.

10 pts. total

2. The pth quantile of the standard normal distribution is the constant z_p that satisfies $P(Z \le z_p) = p$, where X is normally distributed with a mean of zero and variance of one.

5pts.

(a) Circle the answer closest to the true value of $z_{0.98}$.

-2 -1 0 1 2

5 pts.

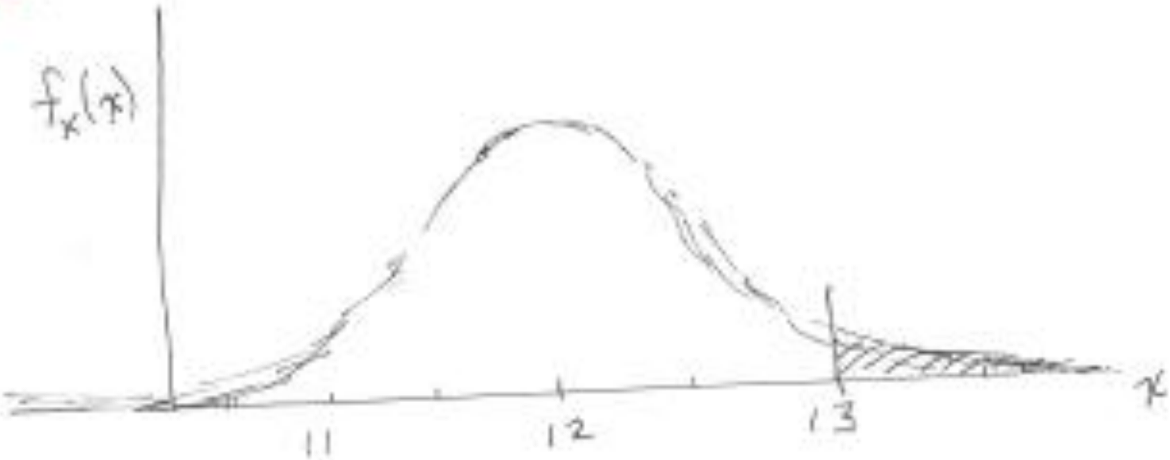
(b) Recall that the normal density is symmetric. Circle the true statement.

$z_p = 1 - z_{1-p}$. $z_p = -z_{1-p}$. $z_p = 2\, z_{p/2}$.

15 pts. total

3. (From Montgomery and Runger, 5.53.) Assume that the weight, X, of a particular kind of running shoe is normally distributed with mean 12 ounces and standard deviation 0.5 ounce.

5 pts. (a) Sketch f_X. Label both axes. Scale the horizontal axis.



5 pts. (b) On the sketch show the probability that a randomly selected shoe weighs more than 13 ounces.		See shaded area.

5 pts. (c) If the standard deviation remains at 0.5 ounce, what must the mean weight μ be so that 99.9% of its shoes are less than 13 ounces. (Information: The standard normal quantile value is $z_{.999} = 3.09$.)

$$P(X < 13) = .999$$

$$\Rightarrow P\left(\frac{X-\mu}{\sigma} < \frac{13-\mu}{\sigma}\right) = .999$$

$$\Rightarrow P\left(Z < \frac{13-\mu}{.5}\right) = .999$$

$$\Rightarrow \frac{13-\mu}{.5} = z_{.999} = 3.09$$

$$\Rightarrow \mu = 13-(.5)(3.09)$$

$$= 11.455 \text{ oz.}$$

15pts. total

4. My car has a 16-gallon gas tank. When I buy gasoline, I fill the tank completely.

Let X denote the gasoline-gauge reading when I fill the tank. I never run out of gasoline; the gauge reading when I fill up is uniform between 0 and 1/4.

Let Y denote my mileage (in miles per gallon) for a randomly selected tank of gasoline. Assume that the distribution of Y is gamma with mean 22 and standard deviation 3. Assume that X and Y are independent.

5 pts. (a) Write the joint density function of (X, Y). (Include all values of x and y.)

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) \quad \text{since } X \text{ and } Y \text{ are independent}$$

$$= (4)\,\frac{\lambda^r y^{r-1} e^{-\lambda y}}{\Gamma(r)} \quad \text{if } \begin{cases} 0 \le x \le 1/4 \\ 0 < y < \infty \end{cases}$$

and zero elsewhere

where

$$\left.\begin{aligned} r/\lambda &= 22 \\ \sqrt{r/\lambda^2} &= 3 \end{aligned}\right\} \Rightarrow$$

5 pts. (b) Suppose that, because of much highway driving, $Y = 28.2$. What is the conditional mean gauge reading, $\mu_{X\,|\,28.2}$.

$$\mu_{X|28.2} = \mu_X \quad \text{because } X \text{ and } Y \text{ are independent}$$

$$= \frac{0 + 1/4}{2} = \frac{1}{8} \quad \text{because } X \text{ is } U(0, 1/4).$$

5 pts. (c) Let M denote the number of miles driven on a (randomly selected) tank. Then $M = 16\,(1-X)\,Y$. Is M independent of X? Yes No

10 pts. total

5. Suppose that (X, Y) is a continuous random vector with cdf $F(x, y) = 4xy$ for all $a \le x \le b$ and $a \le y \le b$ (and zero density elsewhere).

5 pts. (a) What is the value of a?

$$0 = F(a,a) \Rightarrow 4a^2 \Rightarrow a = 0 \leftarrow$$

5 pts. (b) What is the value of b?

$$1 = F(b,b) = 4b^2 \Rightarrow b = 1/2 \leftarrow$$

25 pts. total

6. Consider defects that occur as we manufacture copper tubing. Assume that the defects occur according to a Poisson process with rate $\lambda = 0.02$ defects per foot. Let T_j denote the location of the jth defect as measured (in feet) from the beginning of this run of tubing.

Suppose that the tubing is cut into tubes eight-feet long. Consider a pile of ten tubes. Let X_i denote the number of defects in tube i for $i = 1, 2, \ldots, 10$.

Tube i is classified as being "defect free" if $X_i = 0$, "good" if $1 \le X_i \le 2$, and "defective" if $3 \le X_i$.

Still considering the ten tubes, let Y_1 denote the number of "defect free" tubes, Y_2 the number of "good" tubes.

5 pts. (a) What is the value of $E(X_1)$?

$X_1 \sim$ Poisson

$E(X_1) = \lambda t = \left(.02 \frac{\text{defects}}{\text{foot}}\right)\left(8 \text{ feet}\right) = 0.16$ defects

5 pts. (b) Let A denote the event that a randomly chosen tube is "defect free". What is the value of P(A)?

$P(A) = P(X_i = 0)$

$= \frac{e^{-.16}(.16)^0}{0!}$

or $P(A) = P(T_1 > 8)$

$= \int_8^\infty (.02) e^{-.02t}\, dt$

$\approx .852$

The fact that the sixth tube is "defect free" ($X_6=0$) or not ($X_6 \neq 0$) will affect the value of Y_1 (number of "defect free" tubes).

5 pts. (c) What is the value of $E(T_3)$?

$T_3 \sim \text{Erlang}(r = 3, \lambda = 0.02)$

$\Rightarrow E(T_3) = \frac{r}{\lambda} = \frac{3 \text{ defects}}{.02 \frac{\text{defects}}{\text{foot}}} = 150$ feet

5 pts. (d) Is Y_1 independent of Y_2? Yes No ($Y_1 = 10 \Rightarrow Y_2 = 0$)

5 pts. (e) Is Y_1 independent of X_6? Yes No ($Y_1 = 1000$ feet $\Rightarrow X_6 = 0$)

Continuous Distributions: Summary Table

random variable	distribution name	range	cumul. dist. func.	probability density func.	expected value	variance
X	general	$(-\infty, \infty)$	$P(X \le x)$ $= F(x)$ $= F_X(x)$	$\left.\frac{dF(y)}{dy}\right\|_{y=x}$ $= f(x)$ $= f_X(x)$	$\int_{-\infty}^{\infty} x f(x) dx$ $= \mu = \mu_X$ $= E(X)$	$\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	continuous uniform	$[a, b]$	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
sum of random variables	normal (or Gaussian)	$(-\infty, \infty)$	Table II	$\frac{e^{\frac{-1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}}{\sqrt{2\pi}\,\sigma}$	μ	σ^2
time to Poisson count	exponential	$[0, \infty)$	$1 - e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
time to rth Poisson count	Erlang	$[0, \infty)$	$\sum_{k=r}^{\infty} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$	r/λ	r/λ^2
lifetime	gamma	$[0, \infty)$	numerical	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$	r/λ	r/λ^2
lifetime	Weibull	$[0, \infty)$	$1 - e^{-(x/\delta)^\beta}$	$\frac{\beta x^{\beta-1} e^{-(x/\delta)^\beta}}{\delta^\beta}$	$\delta\Gamma(1+\frac{1}{\beta})$	$\delta^2\Gamma(1+\frac{2}{\beta})-\mu^2$