IE 230 Seat # ________ Name ___ < KEY > ___

Closed book and notes. No calculators.

60 minutes, but essentially unlimited time.

Cover page, four pages of exam, and Pages 8 and 12 of the Concise Notes.

This test covers through Section 4.7 of Montgomery and Runger, fourth edition.

Reminder: A statement is true only if it is always true.

Score ____________________________

1. True or false. (For each: three points if correct, zero points if blank or wrong.)

   (a) T ← F A random variable is a function that assigns a real number to every outcome in the sample space.

   (b) T ← F If X is a discrete random variable, then $f_X(c) = \mathrm{P}(X = c)$ for every real number c.

   (c) T F ← If X is a continuous random variable, then $f_X(c) = \mathrm{P}(X = c)$ for every real number c.

   (d) T ← F If X is normally distributed with mean 6 and standard deviation 2.3, then $\mathrm{P}(X \le 6) = 0.5$.

   (e) T ← F If X is (continuous) uniformly distributed with mean 6 and standard deviation 2.3, then $\mathrm{P}(X \le 6) = 0.5$.

   (f) T F ← Always, the expected value of a random variable is one of its possible values.

   (g) T ← F If X is a geometric random variable, then $\mathrm{P}(X = 3) = \mathrm{P}(2.5 \le X \le 3.5)$.

   (h) T ← F If X is binomially distributed with mean $\mu_X = 10$ and $n = 25$, then the probability of success is $p = 10/25$.

   (i) T ← F Bernoulli trials are independent of each other.

2. (8 points) (Montgomery and Runger, 3–109) Suppose that X has a Poisson distribution with $\mathrm{P}(X = 0) = 0.06$. Determine the value of $\mathrm{E}(X)$.

The Poisson pmf is $f_X(x) = \mathrm{e}^{-\mu}\mu^x / x!$ for $x = 0, 1, \ldots$.

Therefore,

$$\mathrm{P}(X = 0) = \mathrm{e}^{-\mu}\mu^0 / 0! = \mathrm{e}^{-\mu}.$$

The condition that $\mathrm{P}(X = 0) = 0.06$ yields

$$\mu = -\ln(0.06) \approx 2.81 \leftarrow$$

---

3. (Montgomery and Runger, 3–87) Assume that each of your calls to a popular radio station has a probability of 0.07 of connecting, that is, of not obtaining a busy signal. Assume that your calls are independent.

(a) (6 points) What is the experiment for Part (b)?

Call the station until a call is connected.

(b) (6 points) Determine the probability that your first call that connects is the seventh.

Let X denote the number of calls until the first connected call.

Then X is geometric with probability of success $p = 0.07$.

Therefore,

$$P(X = 7) = f_X(7) = p\,(1-p)^6 = (0.07)\,(0.93)^6 \approx 0.0453 \leftarrow$$

(c) (6 points) Determine the expected number of calls until your first connection.

Because X is geometric,

$$E(X) = 1/p = 1/(0.07) \approx 14.3 \leftarrow$$

4. (7 points) (Montgomery and Runger, 4–13) The gap width is an important property of a magnetic recording head. Suppose that, in coded units, the width is a continuous random variable having the range $0 < x < 2$ with $f_X(x) = x/2$. Determine the cdf F_X.

For every real number x, the cdf is $F_X(x) = \mathrm{P}(X \le x)$. Therefore,

$F_X(x) = 0$ if $x < 0$
$F_X(x) = \int_0^x (c/2)dc = x^2/4$ if $0 < x < 2$
$F_X(x) = 1$ if $x > 2$

---

5. (Montgomery and Runger, 4–54) The fill volume of an automated filling machine used to fill cans of carbonated beverage is normally distributed with a mean of 12.4 fluid ounces and a standard deviation of 0.1.

(a) (4 points) What are the units of the standard deviation?

---

fluid ounces ←

(b) (6 points) Sketch the pdf of X. Label both axes. Scale the horizontal axis.

Sketch a horizontal and a vertical axis.
Sketch the bell curve, which should be symmetric, with the bell disappearing just beyond three times the points of inflection.
Label the horizontal axis with a dummy variable, such as x.
Label the vertical axis with $f_X(x)$.
Scale the horizontal axis with at least two points.
Probably one of the points is the mean.
The points of inflection should be at 12.3 and 12.5 fluid ounces.
The bell should disappear at about 12.1 and 12.7 fluid ounces.

(c) (6 points) On your sketch, indicate the probability that the fill volume is more than 12 fluid ounces.

Shade under the bell curve everything to the right of 12 ounces.
All (or almost all) of the visible area should be shaded.

(d) (6 points) Determine the value (approximately) of the probability that the fill volume is more than 12 fluid ounces.

12.0 ounces is four standard deviations below the mean.
We know that the area within three standard deviations from the mean is about 0.997.
Therefore, the area greater than 12.1 ounces is about 0.9985.
Therefore, the area greater than 12.0 ounces is between 0.9985 and 1.
A reasonable guess might be $\mathrm{P}(X < 12.0) = 0.9999$ ←
(The correct value, from Table III or MSexcel or ..., is closer to 0.99997.)

---

6. (Montgomery and Runger, 4–73) Suppose that the number of asbestos particles in a sample of one squared centimeter of dust is a Poisson random variable with a mean of 1000.

(a) (6 points) Determine the Poisson-process rate of the asbestos particles. (State the units.)

$\lambda = 1000$ (particles per squared centimeter) $\leftarrow$

(b) (6 points) Determine the probability that no particles are found in two square centimeters of dust.

Let X denote the number of particles in two square centimeters.

Then X is Poisson with mean $\mu = \lambda t = (1000)(2) = 2000$ particles.

Therefore, $P(X = x) = f_X(x) = e^{-\mu}\mu^x / x!$ for $x = 0, 1, \ldots$.

Therefore, $P(X = 0) = e^{-2000}\, 2000^0 / 0! = e^{-2000} \leftarrow$

(c) (6 points) Consider ten squared centimeters of dust. Determine the expected number of asbestos particles.

Let Y denote the number of particles in ten square centimeters.

Then X is Poisson with mean $\mu = \lambda t = (1000)(10) = 10000$ particles. $\leftarrow$

Discrete Distributions: Summary Table

random variable	distribution name	range	probability mass function	expected value	variance
X	general	$x_1, x_2, \ldots, x_n$	$P(X = x)$ $= f(x)$ $= f_X(x)$	$\sum_{i=1}^{n} x_i f(x_i)$ $= \mu = \mu_X$ $= E(X)$	$\sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	discrete uniform	$x_1, x_2, \ldots, x_n$	$1/n$	$\sum_{i=1}^{n} x_i / n$	$[\sum_{i=1}^{n} x_i^2 / n] - \mu^2$
X	equal-space uniform	$x = a, a+c, \ldots, b$ where	$1/n$ $n = (b-a+c)/c$	$\frac{a+b}{2}$	$\frac{c^2(n^2-1)}{12}$
	indicator variable	$x = 0, 1$	$p^x (1-p)^{1-x}$	p where	$p(1-p)$ $p = P(\text{"success"})$
	binomial	$x = 0, 1, \ldots, n$	$C_x^n p^x (1-p)^{n-x}$	np where	$np(1-p)$ $p = P(\text{"success"})$
	hyper-geometric	$x =$ $(n-(N-K))^+, \ldots, \min\{K, n\}$ and integer	$C_x^K C_{n-x}^{N-K} / C_n^N$	np where	$np(1-p)\frac{(N-n)}{(N-1)}$ $p = K/N$
	geometric	$x = 1, 2, \ldots$	$p(1-p)^{x-1}$	$1/p$ where	$(1-p)/p^2$ $p = P(\text{"success"})$
	negative binomial	$x = r, r+1, \ldots$	$C_{r-1}^{x-1} p^r (1-p)^{x-r}$	r/p where	$r(1-p)/p^2$ $p = P(\text{"success"})$
	Poisson	$x = 0, 1, \ldots$	$e^{-\mu} \mu^x / x!$	μ where	μ $\mu = \lambda t$

Result. For $x = 1, 2, \ldots$, the geometric cdf is $F_X(x) = 1 - (1-p)^x$.

Result. The geometric distribution is the only discrete memoryless distribution. That is, $P(X > x + c \mid X > x) = P(X > c)$.

Result. The binomial distribution with $p = K/N$ is a good approximation to the hypergeometric distribution when n is small compared to N.

Continuous Distributions: Summary Table

random variable	distribution name	range	cumulative distrib. func.	probability density func.	expected value	variance
X	general	$(-\infty, \infty)$	$P(X \le x)$ $= F(x)$ $= F_X(x)$	$\left.\frac{dF(y)}{dy}\right\|_{y=x}$ $= f(x)$ $= f_X(x)$	$\int_{-\infty}^{\infty} x f(x) dx$ $= \mu = \mu_X$ $= E(X)$	$\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	continuous uniform	$[a, b]$	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
X	triangular	$[a, b]$	$(x-a)f(x)/2$ if $x \le m$, else $1-(b-x)f(x)/2$	$\frac{2(x-d)}{(b-a)(m-d)}$ ($d = a$ if $x \le m$, else $d = b$)	$\frac{a+m+b}{3}$	$\frac{(b-a)^2-(m-a)(b-m)}{18}$
sum of random variables	normal (or Gaussian)	$(-\infty, \infty)$	Table III	$\frac{e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}$	μ	σ^2
time to Poisson count 1	exponential	$[0, \infty)$	$1 - e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
time to Poisson count r	Erlang	$[0, \infty)$	$\sum_{k=r}^{\infty} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$	r/λ	r/λ^2
lifetime	gamma	$[0, \infty)$	numerical	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$	r/λ	r/λ^2
lifetime	Weibull	$[0, \infty)$	$1 - e^{-(x/\delta)^\beta}$	$\frac{\beta x^{\beta-1} e^{-(x/\delta)^\beta}}{\delta^\beta}$	$\delta\Gamma(1+\frac{1}{\beta})$	$\delta^2\Gamma(1+\frac{2}{\beta})-\mu^2$

Definition. For any $r > 0$, the *gamma function* is $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx$.

Result. $\Gamma(r) = (r-1)\Gamma(r-1)$. In particular, if r is a positive integer, then $\Gamma(r) = (r-1)!$.

Result. The exponential distribution is the only continuous memoryless distribution.
That is, $P(X > x + c \mid X > x) = P(X > c)$.

Definition. A *lifetime* distribution is continuous with range $[0, \infty)$.

Modeling lifetimes. Some useful lifetime distributions are the exponential, Erlang, gamma, and Weibull.