IE 230    Seat # ________                                        Name ___ < KEY > ___

**Please read these directions.**

*Closed book and notes.  60 minutes.*

Covers through the normal distribution, Section 4.6 of Montgomery and Runger, fourth edition.

Cover page and four pages of exam.
Pages 8 and 12 of the Concise Notes.
A normal-distribution cdf table.

No calculator.  No need to simplify beyond probability concepts.
For example, unsimplified factorials, integrals, sums, and algebra receive full credit.

Throughout, $f$ denotes probability mass function or probability density function
and $F$ denotes cumulative distribution function.

For one point of credit, write your name neatly on this cover page.  Circle your family name.

For one point of credit, write your name on each of pages 1 through 4.

Score ____________________________

*Closed book and notes. 60 minutes.*

1. Suppose that the random variable $X$ has the discrete uniform distribution over the set $\{0, 1, ..., 10\}$ and that the random variable $Y$ has the continuous uniform distribution over the set $[0, 10]$.

   (a) (3 pt) T ← F $\text{E}(X) = \text{E}(Y)$.

   (b) (3 pts) T F ← $\text{V}(X) = \text{V}(Y)$.

   (c) (3 pts) T F ← $F_X(5) = F_Y(5)$.

   (d) (3 pts) T F ← $f_X(5) = f_Y(5)$.

2. (from Montgomery and Runger, 3–113) In 1898 L. J. Bortkiewicz published a book entitled *The Law of Small Numbers*. He used data collected over twenty years to show that the number of soldiers killed by horse kicks each year in each corps in the Prussian cavalry followed a Poisson distribution with a mean of 0.61.

   (a) (7 pts) For a particular corps over one year, determine the value of the probability of more than one death.

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Deaths occur with rate $\lambda = 0.61$ deaths per year per corps.

Let $N$ = "number of deaths in one year in a particular corps".

Then $N$ is Poisson with rate $\mu = \lambda t = (0.61)(1) = 0.61$ deaths.

Therefore, $\text{P}(N > 1) = 1 - [\text{P}(N = 0) + \text{P}(N = 1)]$

$$= 1 - \left[\frac{e^{-\mu}\mu^0}{0!} + \frac{e^{-\mu}\mu^1}{1!}\right] = 1 - e^{-0.61}[1 + 0.61] \leftarrow$$

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   (b) (7 pts) For a particular corps over five years, determine the value of the probability of no death.

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Deaths still occur with rate $\lambda = 0.61$ deaths per year per corps.

Now let $N$ denote the number of deaths during five years.

Then $N$ is Poisson with mean $\mu = \lambda t = (0.61)(5) = 3.05$.

Therefore, $\text{P}(N = 0) = \dfrac{e^{-\mu}\mu^0}{0!} = e^{-3.05} \leftarrow$

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3. (from Montgomery and Runger, 4–61) The lifetime of a semiconductor laser at a constant power is normally distributed with a mean of 6000 hours and standard deviation of 500 hours.

(a) (8 pts) Sketch the corresponding normal pdf. Label and scale both axes.

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Sketch two axes.
Label the horizontal axis with $x$ and the vertical with $f_X(x)$.
Scale the horizontal axis with at least two numbers.
The easiest is to place the mean 6000 at the center.
Maybe place 5500 and 6500 at the points of inflection.
Maybe place 4500 and 7500 where the bell curve disappears.
Scale the vertical axis.
Placing zero at the bottom of the bell curve is an easy number.
The height at the mode is $1 / (500\sqrt{2\pi})$.
Or sketch a rectangle to determine a height.

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(b) (8 pts) Determine the value of the probability that a randomly selected laser fails between 4000 and 5000 hours. Provide at least three digits of precision.

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$$\mathrm{P}(4000 < X < 5000) = \mathrm{P}(7000 < X < 8000) = F_X(8000) - F_X(7000)$$
$$= F_Z(4) - F_Z(2) \approx 0.999968 - 0.977250 \approx 0.0227 \leftarrow$$

Notice that including, or excluding, 4000 and 5000 makes no difference.

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(c) (8 pts) Let $q$ denote the correct answer to Part (b). If three lasers are used in a product, and all have independent lifetimes, determine the value of the probability that all three lasers fail between 4000 and 5000 hours.

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Let $L_i$ denote that laser $i$ has a lifetime in $(4000, 5000)$.

Then $\mathrm{P}(L_i) = q \approx 0.0227$.

Then $L_1$, $L_2$, and $L_3$ are Bernoulli trials, with probability of success $q$.

Let $X$ denote the number of trials that succeed.

So $X$ is binomial with $n = 3$ and $p = q$. From Page 8 of the Concise Notes,

$$\mathrm{P}(N = 3) = f_X(3) = \binom{3}{3} q^3 (1-q)^{3-3} = q^3 \leftarrow$$

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4. Result. For $c = 1, 2, ...,$ the geometric cdf is $F_X(c) = 1 - (1-p)^c$, where $p$ denotes the probability of success on each Bernoulli trial.

(3 pts each) For each of the six lines (a–f), state why the corresponding equality is true.

Each blank requires one reason; reasons may be reused. Choose the reasons from this list:

(i) Events partition the sample space.

(ii) Events are complementary.

(iii) Events are mutually exclusive.

(iv) Events are independent.

(v) Events are the same.

(vi) Definition of conditional probability.

(vii) Definition of cumulative distribution function.

(viii) Definition of probability mass function.

(ix) Multiplication Rule.

(x) Total Probability.

(xi) Substitute known values.

Proof. Let $c$ be a positive integer. Let $A_i$ denote success on the $i$th trial.

| | | |
|---|---|---|
| $F_X(c)$ | $= P(X \le c)$ | (a) ___ < vii > ___ |
| | $= 1 - P(X > c)$ | (b) ___ < ii > ___ |
| | $= 1 - P(A'_1 \cap A'_2 \cap \cdots \cap A'_c)$ | (c) ___ < v > ___ |
| | $= 1 - P(A'_1)\,P(A'_2) \cdots P(A'_c)$ | (d) ___ < iv > ___ |
| | $= 1 - [1 - P(A_1)]\,[1 - P(A_2)] \cdots [1 - P(A_c)]$ | (e) ___ < ii > ___ |
| | $= 1 - (1-p)^c$ | (f) ___ < xi > ___ |

5. For (a–e), provide the name of the corresponding family of distributions.

(a) (3 pts) The number of coin flips until the fourth "head".

negative binomial

(b) (3 pts) When drawing, without replacement, five cards from a standard card deck, the number of aces drawn.

hypergeometric ←

(c) (3 pts) When drawing, with replacement, five cards from a standard card deck, the number of aces drawn.

binomial ←

(d) (3 pts) The number of insect parts in a candy bar.

Poisson ←

(e) (3 pts) When playing Keno, the number of matched numbers.

hypergeometric ←

6. Short answer.

(a) (3 pts) Evaluate $\binom{6}{2}$, the number of ways to choose two items from six items.

$\binom{6}{2} = \dfrac{6!}{2!\,4!} = 15$ ←

(b) (3 pts) What is the numerical value of $0!$ ?

$1! = 1$ ←

(c) (3 pts) T ← F $\quad \mathrm{P}(\pi < X \le 2\pi) = F_X(2\pi) - F_X(\pi)$.

(d) (3 pts) The mean of the binomial distribution is $n\,p$, the number of trials multiplied by the probability of success. The mean is in units of "trials".
What are the units of the variance?

"trials" squared ←

(e) (3 pts) The mean of a distribution corresponds to the center of gravity.
The variance of a distribution corresponds to...

the moment of inertia. ←

Discrete Distributions: Summary Table

| random variable | distribution name | range | probability mass function | expected value | variance |
|---|---|---|---|---|---|
| $X$ | general | $x_1, x_2, \ldots, x_n$ | $P(X = x)$ $= f(x)$ $= f_X(x)$ | $\sum_{i=1}^{n} x_i f(x_i)$ $= \mu = \mu_X$ $= E(X)$ | $\sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$ |
| $X$ | discrete uniform | $x_1, x_2, \ldots, x_n$ | $1/n$ | $\sum_{i=1}^{n} x_i / n$ | $[\sum_{i=1}^{n} x_i^2 / n] - \mu^2$ |
| $X$ | equal-space uniform | $x = a, a+c, \ldots, b$ where | $1/n$ $n = (b-a+c)/c$ | $\frac{a+b}{2}$ | $\frac{c^2(n^2-1)}{12}$ |
| "# successes in 1 Bernoulli trial" | indicator variable | $x = 0, 1$ | $p^x (1-p)^{1-x}$ | $p$ where | $p(1-p)$ $p = P(\text{"success"})$ |
| "# successes in $n$ Bernoulli trials" | binomial | $x = 0, 1, \ldots, n$ | $C_x^n p^x (1-p)^{n-x}$ | $np$ where | $np(1-p)$ $p = P(\text{"success"})$ |
| "# successes in a sample of size $n$ from a population of size $N$ containing $K$ successes" | hyper-geometric (sampling without replacement) | $x =$ $(n-(N-K))^+$, ..., $\min\{K, n\}$ and integer | $C_x^K C_{n-x}^{N-K} / C_n^N$ | $np$ where | $np(1-p)\frac{(N-n)}{(N-1)}$ $p = K/N$ |
| "# Bernoulli trials until 1st success" | geometric | $x = 1, 2, \ldots$ | $p(1-p)^{x-1}$ | $1/p$ where | $(1-p)/p^2$ $p = P(\text{"success"})$ |
| "# Bernoulli trials until $r$th success" | negative binomial | $x = r, r+1, \ldots$ | $C_{r-1}^{x-1} p^r (1-p)^{x-r}$ | $r/p$ where | $r(1-p)/p^2$ $p = P(\text{"success"})$ |
| "# of counts in time $t$ from a Poisson process with rate $\lambda$" | Poisson | $x = 0, 1, \ldots$ | $e^{-\mu} \mu^x / x!$ | $\mu$ where | $\mu$ $\mu = \lambda t$ |

Result. For $x = 1, 2, \ldots$, the geometric cdf is $F_X(x) = 1 - (1-p)^x$.

Result. The geometric distribution is the only discrete memoryless distribution.
That is, $P(X > x + c \mid X > x) = P(X > c)$.

Result. The binomial distribution with $p = K/N$ is a good approximation to the hypergeometric distribution when $n$ is small compared to $N$.

Continuous Distributions: Summary Table

| random variable | distribution name | range | cumulative distrib. func. | probability density func. | expected value | variance |
|---|---|---|---|---|---|---|
| $X$ | general | $(-\infty,\infty)$ | $P(X \le x)$ $= F(x)$ $= F_X(x)$ | $\left.\frac{dF(y)}{dy}\right\rvert_{y=x}$ $= f(x)$ $= f_X(x)$ | $\int_{-\infty}^{\infty} x f(x) dx$ $= \mu = \mu_X$ $= E(X)$ | $\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$ |
| $X$ | continuous uniform | $[a, b]$ | $\frac{x-a}{b-a}$ | $\frac{1}{b-a}$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ |
| $X$ | triangular | $[a, b]$ | $(x-a)f(x)/2$ if $x \le m$, else $1-(b-x)f(x)/2$ | $\frac{2(x-d)}{(b-a)(m-d)}$ ($d = a$ if $x \le m$, | $\frac{a+m+b}{3}$ else $d = b$) | $\frac{(b-a)^2-(m-a)(b-m)}{18}$ |
| sum of random variables | normal (or Gaussian) | $(-\infty,\infty)$ | Table III | $\frac{e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}$ | $\mu$ | $\sigma^2$ |
| time to Poisson count 1 | exponential | $[0,\infty)$ | $1 - e^{-\lambda x}$ | $\lambda e^{-\lambda x}$ | $1/\lambda$ | $1/\lambda^2$ |
| time to Poisson count $r$ | Erlang | $[0,\infty)$ | $\sum_{k=r}^{\infty} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$ | $\frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$ | $r/\lambda$ | $r/\lambda^2$ |
| lifetime | gamma | $[0,\infty)$ | numerical | $\frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$ | $r/\lambda$ | $r/\lambda^2$ |
| lifetime | Weibull | $[0,\infty)$ | $1 - e^{-(x/\delta)^\beta}$ | $\frac{\beta x^{\beta-1} e^{-(x/\delta)^\beta}}{\delta^\beta}$ | $\delta\Gamma(1+\frac{1}{\beta})$ | $\delta^2\Gamma(1+\frac{2}{\beta})-\mu^2$ |

Definition. For any $r > 0$, the *gamma function* is $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx$.

Result. $\Gamma(r) = (r-1)\Gamma(r-1)$. In particular, if $r$ is a positive integer, then $\Gamma(r) = (r-1)!$.

Result. The exponential distribution is the only continuous memoryless distribution.
That is, $P(X > x + c \mid X > x) = P(X > c)$.

Definition. A *lifetime* distribution is continuous with range $[0,\infty)$.

Modeling lifetimes. Some useful lifetime distributions are the exponential, Erlang, gamma, and Weibull.