Closed book and notes. No calculators.

60 minutes, but essentially unlimited time.

Cover page, four pages of exam, and Pages 8 and 12 of the Concise Notes.

This test covers through Section 4.5 of Montgomery and Runger, fourth edition.

Reminder: A statement is true only if it is always true.

Recall: If X is the indicator function of the event A, then $X = 1$ if A occurs and $X = 0$ if A does not occur.

Score ____________________________

1. True or false. (For each: three points if correct, zero points if blank or wrong.)

(a) T F ← The mean of a Poisson distribution is always a nonnegative integer.

(b) T F ← If X is a random variable, then $f_X(6) = \mathrm{P}(X = 6)$.

(c) T F ← If X is a continuous random variable, then $f_X(6) \le 1$.

(d) T ← F If X is a continuous random variable, the $\mathrm{P}(X = 6) = 0$.

(e) T ← F Let X denote a random variable whose smallest value is a and largest value is b. Then always $a \le \mathrm{E}(X) \le b$.

(f) T ← F If the random variable X has units "US dollars", then its expected value is also in "US dollars" and its standard deviation is in "US dollars".

(g) T ← F For every random variable X, $\mathrm{P}(6 < X \le 10) = \mathrm{F}_X(10) - F_X(6)$.

(h) T ← F If X is the indicator function for the event A, then $\mathrm{E}(X) = \mathrm{P}(A)$.

(i) T ← F If X is a continuous random variable, then $\mathrm{P}(a \le X \le b) = \mathrm{P}(a < X < b)$.

2. Short answer (four points each)

a. Define *random variable*.

A function that assigns a real number to every outcome in the sample space.

b. Define *Bernoulli trial*.

Bernoulli trials have exactly two outcomes, usually denoted by "success" and "failure". The probabilities of success and failure, usually denoted by p and $1 - p$, are constant. The trials are independent of each other.

c. Determine the value of C_7^9, the number of ways to take seven items from a set of nine items. (Simplify)

$C_7^9 = 9! / (7!\, 2!) = (9)(8) / 2 = 36$ ←

3. (Montgomery and Runger, 3–119) Let the random variable Y be equally likely to assume any of the values $1/8$, $1/4$, or $5/8$. No other values are possible.

(a) (five points) Write f_Y, the probably mass function of Y.

$f_Y(c) = 1/3$ if $c = 1/8$, $1/4$, or $5/8$.
Otherwise $f_Y(c) = 0$.

(b) (five points) Determine, $\mathrm{E}(Y)$, the expected value of Y.

$\mathrm{E}(Y) = (1/8)(1/3) + (1/4)(1/3) + (5/8)(1/3) = 1/3 \leftarrow$

(c) (four points) Determine $F_Y(0.5)$, the cumulative distribution function evaluated at 0.5.

$F_Y(0.5) = \mathrm{P}(Y \le 0.5) = \mathrm{P}(Y = 1/8) + \mathrm{P}(Y = 1/4) = f_Y(1/8) + f_Y(1/4) = 2/3 \leftarrow$

4. Let X denote the number of matches in Keno, where 20 of 80 numbers light up. Suppose that you, the player, have chosen 5 numbers.

(a) (three points) Write the values for the hypergeometric parameters N, K, and n.

$N = 80 \leftarrow$
$K = 20 \leftarrow$
$n = 5 \leftarrow$

(b) (four points) Determine the probability of more than one match.

P(more than one match) $= \mathrm{P}(X > 1) = f_X(2) + f_X(3) + f_X(4) + f_X(5)$

or

P(more than one match) $= \mathrm{P}(X > 1) = 1 - [f_X(0) + f_X(1)]$,

where

$f_X(c) = C_c^{20}\, C_{20-c}^{60} \,/\, C_5^{80}$.

5. (Montgomery and Runger, 3–121) Batches that consist of 52 coil springs from a production process are checked for conformance to customer requirements. The mean number of nonconforming coil springs in a batch is five. Assume that the number of nonconforming springs in a batch, denoted by X, is a binomial random variable.

(a) (four points) What is the experiment?

Choose a random batch of 52 coil springs.

(b) (two points) What is the value of the parameter n? ___ < 52 > ___

(c) (two points) What is the value of the parameter p? ____< 5 / 52 > ___

(d) (six points) Determine the probability of at least 51 nonconforming springs.

P(at least 51 nonconforming springs) = $\text{P}(X \geq 51) = f_X(51) + f_X(52)$
where
$f_X(c) = C_c^n \, p^c \, (1-p)^{n-c}$

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6. (Montgomery and Runger, 4–5) Consider the pdf $f_X(x) = cx^2$ for $-1 \leq x \leq 1$.

(a) (five points) Sketch f_X, including labeling and scaling both axes.

Sketch a horizontal and a vertical axis.
Label the horizontal axis with a dummy variable, such as x.
Label the veritical axis with $f_X(x)$.
Scale the horizontal axis with at least two numbers, probably −1, 0, and 1.
Scale the vertical axis with at least two numbers, probably 0 and $c = 3/2$.
Sketch the function f_X, including the zero values outside the interval [−1, 1].

Comment: $c = 3/2$ is the value that yields one as the area under f_X.

(b) (five points) Sketch the cdf F_X, including labeling and scaling both axes.

Sketch a horizontal and a vertical axis.
Label the horizontal axis with a dummy variable, such as x.
Label the veritical axis with $F_X(x)$.
Scale the horizontal axis with at least two numbers, probably −1, 0, and 1.
Scale the vertical axis with at least two numbers, probably 0 and 1.
Sketch the function $F_X(x) = \text{P}(X \leq x) = (x^3 + 1)/2$ when $-1 \leq x \leq 1$,
$F_X(x) = 0$ when $x < -1$, and $F_X(x) = 1$ when $x > 1$.

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7. Result: If X is a geometric random variable, then $F_X(x) \le 1 - (1-p)^x$ for $x = 1, 2, \ldots$, where $1-p$ denotes the probability of failure.

   (4 points each) For each of the four lines, state why the corresponding equality is true.

   Let A_i denote the event that the ith Bernoulli trial is a failure.

   Each blank requires one reason; reasons may be reused. Choose the reasons from this list:

   (i) Events partition the sample space.

   (ii) Events are complementary.

   (iii) Events are mutually exclusive.

   (iv) Events are independent.

   (v) Events are the same.

   (vi) Definition of conditional probability.

   (vii) Multiplication Rule.

   (viii) Total Probability.

   (ix) Bayes's Rule.

   (x) Algebra (i.e., no set theory or probability needed).

   (xi) Definition of probability mass function.

   (xii) Definition of cumulative distribution function.

   (xiii) X is continuous.

   (xiv) X is discrete.

$F_X(x)$	$= \mathrm{P}(X \le x)$	___ < xii > ___
	$= 1 - \mathrm{P}(X > x)$	___ < ii > ___
	$= 1 - \mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_x)$	___ < v > ___
	$= 1 - \mathrm{P}(A_1)\,\mathrm{P}(A_2) \ldots \mathrm{P}(A_x)$	___ < iv > ___
	$= 1 - (1-p)^x$	Definition of $1-p$.

Discrete Distributions: Summary Table

random variable	distribution name	range	probability mass function	expected value	variance
X	general	$x_1, x_2, \ldots, x_n$	$P(X = x)$ $= f(x)$ $= f_X(x)$	$\sum_{i=1}^{n} x_i f(x_i)$ $= \mu = \mu_X$ $= E(X)$	$\sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	discrete uniform	$x_1, x_2, \ldots, x_n$	$1/n$	$\sum_{i=1}^{n} x_i / n$	$[\sum_{i=1}^{n} x_i^2 / n] - \mu^2$
X	equal-space uniform	$x = a, a+c, \ldots, b$ where	$1/n$ $n = (b - a + c)/c$	$\frac{a+b}{2}$	$\frac{c^2(n^2-1)}{12}$
	indicator variable	$x = 0, 1$	$p^x (1-p)^{1-x}$	p where	$p(1-p)$ $p = P(\text{"success"})$
	binomial	$x = 0, 1, \ldots, n$	$C_x^n p^x (1-p)^{n-x}$	np where	$np(1-p)$ $p = P(\text{"success"})$
	hyper-geometric	$x =$ $(n-(N-K))^+$, $\ldots, \min\{K, n\}$ and integer	$C_x^K C_{n-x}^{N-K} / C_n^N$	np where	$np(1-p)\frac{(N-n)}{(N-1)}$ $p = K/N$
	geometric	$x = 1, 2, \ldots$	$p(1-p)^{x-1}$	$1/p$ where	$(1-p)/p^2$ $p = P(\text{"success"})$
	negative binomial	$x = r, r+1, \ldots$	$C_{r-1}^{x-1} p^r (1-p)^{x-r}$	r/p where	$r(1-p)/p^2$ $p = P(\text{"success"})$
	Poisson	$x = 0, 1, \ldots$	$e^{-\mu} \mu^x / x!$	μ where	μ $\mu = \lambda t$

Result. For $x = 1, 2, \ldots$, the geometric cdf is $F_X(x) = 1 - (1 - p)^x$.

Result. The geometric distribution is the only discrete memoryless distribution.
That is, $P(X > x + c \mid X > x) = P(X > c)$.

Result. The binomial distribution with $p = K/N$ is a good approximation to the hypergeometric distribution when n is small compared to N.

Continuous Distributions: Summary Table

random variable	distribution name	range	cumulative distrib. func.	probability density func.	expected value	variance
X	general	$(-\infty, \infty)$	$P(X \le x)$ $= F(x)$ $= F_X(x)$	$\left.\frac{dF(y)}{dy}\right\|_{y=x}$ $= f(x)$ $= f_X(x)$	$\int_{-\infty}^{\infty} x f(x) dx$ $= \mu = \mu_X$ $= E(X)$	$\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	continuous uniform	$[a, b]$	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
X	triangular	$[a, b]$	$(x-a)f(x)/2$ if $x \le m$, else $1-(b-x)f(x)/2$	$\frac{2(x-d)}{(b-a)(m-d)}$ ($d = a$ if $x \le m$, else $d = b$)	$\frac{a+m+b}{3}$	$\frac{(b-a)^2-(m-a)(b-m)}{18}$
sum of random variables	normal (or Gaussian)	$(-\infty, \infty)$	Table III	$\frac{e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}$	μ	σ^2
time to Poisson count 1	exponential	$[0, \infty)$	$1 - e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
time to Poisson count r	Erlang	$[0, \infty)$	$\sum_{k=r}^{\infty} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$	r/λ	r/λ^2
lifetime	gamma	$[0, \infty)$	numerical	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$	r/λ	r/λ^2
lifetime	Weibull	$[0, \infty)$	$1 - e^{-(x/\delta)^\beta}$	$\frac{\beta x^{\beta-1} e^{-(x/\delta)^\beta}}{\delta^\beta}$	$\delta\Gamma(1+\frac{1}{\beta})$	$\delta^2\Gamma(1+\frac{2}{\beta})-\mu^2$

Definition. For any $r > 0$, the *gamma function* is $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx$.

Result. $\Gamma(r) = (r-1)\Gamma(r-1)$. In particular, if r is a positive integer, then $\Gamma(r) = (r-1)!$.

Result. The exponential distribution is the only continuous memoryless distribution. That is, $P(X > x + c \mid X > x) = P(X > c)$.

Definition. A *lifetime* distribution is continuous with range $[0, \infty)$.

Modeling lifetimes. Some useful lifetime distributions are the exponential, Erlang, gamma, and Weibull.