 Seat # ________ (1 pt.) Name ___ < KEY > ___

Closed book and notes. No calculators.

60 minutes, but essentially unlimited time.

Cover page, four pages of exam, and Pages 8 and 12 of the Concise Notes.

This test covers through Section 4.6 of Montgomery and Runger, fourth edition.

Score ____________________________

1. True or false. (For each: three points if correct, zero points if blank or wrong.)

   (a) T F ← The geometric family of distributions is a special case of the binomial family of distributions.

   (b) T F ← The (continuous) uniform family of distributions is a special case of the normal family of distributions.

   (c) T F ← A Bernoulli trial is the special case of the negative-binomial family of distributions when $r = 1$, where r is the number of successes.

   (d) T F ← $0! = 0$.

   (e) T ← F For any random variable X and constant c, the complement of "$X \geq c$" is "$X < c$".

   (f) T F ← If X has a Poisson distribution with a mean of μ, then μ must be a non-negative integer.

   (g) T ← F If f_X is the probability mass function of the random variable X, then $0 \leq f_X(c) \leq 1$ for every value of c.

   (h) T F ← If f_X is the probability density function of the random variable X, then $\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X^2(x)\, dx$.

   (i) T F ← If f_X is the probability density function of the random variable X, then $\mathrm{P}(X = 6) = f_X(6)$.

2. (2 points each) (Montgomery and Runger, 3–88) A player of a video game is confronted with a sequence of opponents and has an 80% chance of defeating each one. Success with any opponent is independent of previous encounters. The player continues to contest opponents until the player is defeated. We are interested in the probability that a player defeats at least two opponents in a game. Let X denote the number of opponents defeated by the player.

   For each blank, insert the reason why the corresponding equal sign is true. An answer can be used more than one time. If more than one answer is correct, choose the answer that is more specific. Choose from these possible answers: mean, standard deviation, same event, complement, mutually exclusive, independence.

   P("player defeats at least two opponents")

$= \mathrm{P}(X \geq 2)$	__ < same event > __
$= 1 - \mathrm{P}(X < 2)$	__ < complement > __
$= 1 - \mathrm{P}(X = 0 \text{ or } X = 1)$	__ < same event > __
$= 1 - [\mathrm{P}(X = 0) + \mathrm{P}(X = 1)]$	__ < mutually exclusive > __
$= 1 - [\mathrm{P}(X = 0) + (1 - 0.80)\, 0.80]$	__ < independence > __

3. Suppose that IQ scores are normally distributed with a mean of 100 and standard deviation of 15.

(a) (5 pts.) Sketch the pdf. Label and scale both axes.

Create horizontal and vertical axes.
Label the horizontal axis with a dummy variable, such as x.
Label the vertical axis with the pdf, such as $f_X(x)$.
Sketch a bell curve, with center at 100 and standard deviation 15.
  The bell curve should be symmetric.
  It should visually disappear at about plus and minus three standard deviations.
  The points of inflection should be one standard deviation from the mean.
  Areas:
    About 68% within one standard deviation from the mean.
    About 95% within two standard deviations from the mean.
    About 99.7% within three standard deviations from the mean.

(b) (5 pts.) Determine the standard-normal z-score that corresponds to an IQ score of 145?

$$z = \frac{x - \mu}{\sigma} = \frac{145 - 100}{15} = 3 \leftarrow$$

(c) (5 pts.) Determine (graphically) the probability that the IQ score is above 100. Show your answer in your sketch of Part (a).

$$P(X > 100) = P\left(\frac{X - \mu}{\sigma} > \frac{100 - 100}{15}\right) = P(Z > 0) = 1/2 \leftarrow$$

Shade the right half of the bell curve in Part (a).

(d) (4 pts.) What is the experiment that underlies Parts (a), (b), and (c)?

Choose a random test taker, or choose a random score.

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4. (6 points) Suppose that a flipped coin has probability p of landing with heads up. Assume that flips are independent. What value of p yields probability 0.95 that four flips yields four heads?

Let X denote the number of of heads in four flips.

Then X is binomial with probability of success p and number of trials $n = 4$.

Then

$$0.95 = \mathrm{P}(X = 4) = C_4^4\ p^4 (1-p)^{4-4} = p^4,$$

which implies that $p = 0.95^{1/4}$ $\leftarrow$

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5. (Montgomery and Runger, 4–9) The probability density function of the length (in feet) of a random metal rod is $f_X(y) = 0.5$ for $3.3 < y < 5.3$ and zero elsewhere.

(a) (5 pts.) Determine the probability that a rod is within 0.1 feet of 5.0 feet long.

$$\mathrm{P}(4.9 < X < 5.1) = \int_{4.9}^{5.1} 0.5\ dc = 0.1 \ \leftarrow$$

(b) (5 pts.) Determine the value of $\mathrm{P}(0.4 \le X \le 5.3)$.

$$\mathrm{P}(0.4 < X < 5.3) = \int_{0.4}^{5.3} f_X(c)\ dc = \int_{3.3}^{5.3} 0.5\ dc = 1 \ \leftarrow$$

(c) (3 pts.) T F $\leftarrow$ The variance of X is

$$\mathrm{V}(X) = (3.3 - 4.3)^2\, 0.5 + (4.3 - 4.3)^2\, 0.5 + (5.3 - 4.3)^2\, 0.5.$$

6. (Montgomery and Runger, 4–5) Suppose that $f_X(c) = 1.5c^2$ for $-1 < c < 1$.

(a) (5 pts.) Determine the value of the cdf $F_X(1.5)$.

$F_X(1.5) = \mathrm{P}(X \le 1.5) = 1 \leftarrow$

(b) (5 pts.) Determine the value of the mean $\mathrm{E}(X)$.

The pdf is symmetric at zero, so $\mathrm{E}(X) = 0 \leftarrow$.

or

$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-1}^{1} x\, 1.5c^2\, dx = 0 \leftarrow$

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7. (Montgomery and Runger, 4–100) In a data-communication system, five messages that arrive at a node are bundled into a packet before they are transmitted over the network. Assume that messages arrive according to a Poisson process with rate $\lambda = 30$ messages per minute.

(a) (5 pts.) Determine the rate that packets are bundled.

λ = (30 messages per minute) (1 packet per 5 messages)

= 6 packets per minute. $\leftarrow$

(b) (5 pts.) Determine the probability that no message arrives during a one-second period.

Let X denote the number of messages that arrive during a one-second period. Then X is Poisson with mean

$\mu = \lambda\, t$

= [(30 messages per minute) (1 minute per 60 seconds)] (one second)

= $1/2$ message.

Therefore, $\mathrm{P}(X = 0) = e^{-\mu}\, \mu^0 / 0! = e^{-1/2} \leftarrow$

Discrete Distributions: Summary Table

random variable	distribution name	range	probability mass function	expected value	variance
X	general	$x_1, x_2, \ldots, x_n$	$P(X = x)$ $= f(x)$ $= f_X(x)$	$\sum_{i=1}^{n} x_i f(x_i)$ $= \mu = \mu_X$ $= E(X)$	$\sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	discrete uniform	$x_1, x_2, \ldots, x_n$	$1/n$	$\sum_{i=1}^{n} x_i / n$	$[\sum_{i=1}^{n} x_i^2 / n] - \mu^2$
X	equal-space uniform	$x = a, a+c, \ldots, b$ where	$1/n$ $n = (b-a+c)/c$	$\frac{a+b}{2}$	$\frac{c^2(n^2-1)}{12}$
	indicator variable	$x = 0, 1$	$p^x(1-p)^{1-x}$	p where	$p(1-p)$ $p = P(\text{"success"})$
	binomial	$x = 0, 1, \ldots, n$	$C_x^n p^x (1-p)^{n-x}$	np where	$np(1-p)$ $p = P(\text{"success"})$
	hyper-geometric	$x =$ $(n-(N-K))^+$, $\ldots, \min\{K, n\}$ and integer	$C_x^K C_{n-x}^{N-K} / C_n^N$	np where	$np(1-p)\frac{(N-n)}{(N-1)}$ $p = K/N$
	geometric	$x = 1, 2, \ldots$	$p(1-p)^{x-1}$	$1/p$ where	$(1-p)/p^2$ $p = P(\text{"success"})$
	negative binomial	$x = r, r+1, \ldots$	$C_{r-1}^{x-1} p^r (1-p)^{x-r}$	r/p where	$r(1-p)/p^2$ $p = P(\text{"success"})$
	Poisson	$x = 0, 1, \ldots$	$e^{-\mu} \mu^x / x!$	μ where	μ $\mu = \lambda t$

Result. For $x = 1, 2, \ldots$, the geometric cdf is $F_X(x) = 1 - (1-p)^x$.

Result. The geometric distribution is the only discrete memoryless distribution. That is, $P(X > x + c \mid X > x) = P(X > c)$.

Result. The binomial distribution with $p = K/N$ is a good approximation to the hypergeometric distribution when n is small compared to N.

Continuous Distributions: Summary Table

random variable	distribution name	range	cumulative distrib. func.	probability density func.	expected value	variance
X	general	$(-\infty, \infty)$	$P(X \le x)$ $= F(x)$ $= F_X(x)$	$\left.\frac{dF(y)}{dy}\right\|_{y=x}$ $= f(x)$ $= f_X(x)$	$\int_{-\infty}^{\infty} x f(x) dx$ $= \mu = \mu_X$ $= E(X)$	$\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	continuous uniform	$[a, b]$	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
X	triangular	$[a, b]$	$(x-a) f(x)/2$ if $x \le m$, else $1-(b-x) f(x)/2$	$\frac{2(x-d)}{(b-a)(m-d)}$ ($d = a$ if $x \le m$, else $d = b$)	$\frac{a+m+b}{3}$	$\frac{(b-a)^2-(m-a)(b-m)}{18}$
sum of random variables	normal (or Gaussian)	$(-\infty, \infty)$	Table II	$\frac{e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}$	μ	σ^2
time to Poisson count 1	exponential	$[0, \infty)$	$1 - e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
time to Poisson count r	Erlang	$[0, \infty)$	$\sum_{k=r}^{\infty} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$	r/λ	r/λ^2
lifetime	gamma	$[0, \infty)$	numerical	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$	r/λ	r/λ^2
lifetime	Weibull	$[0, \infty)$	$1 - e^{-(x/\delta)^\beta}$	$\frac{\beta x^{\beta-1} e^{-(x/\delta)^\beta}}{\delta^\beta}$	$\delta\Gamma(1+\frac{1}{\beta})$	$\delta^2\Gamma(1+\frac{2}{\beta})-\mu^2$

Definition. For any $r > 0$, the *gamma function* is $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx$.

Result. $\Gamma(r) = (r-1)\Gamma(r-1)$. In particular, if r is a positive integer, then $\Gamma(r) = (r-1)!$.

Result. The exponential distribution is the only continuous memoryless distribution. That is, $P(X > x + c \mid X > x) = P(X > c)$.

Definition. A *lifetime* distribution is continuous with range $[0, \infty)$.

Modeling lifetimes. Some useful lifetime distributions are the exponential, Erlang, gamma, and Weibull.