IE 230 Seat # ____________ (1 point) Name (clearly) ___ < KEY > ___

Closed book and notes. No calculators. 60 minutes.

Cover page, four pages of exam, and Page 8 of the Concise Notes.

This test covers through Section 3.6 of Montgomery and Runger, third edition.

Throughout, you will receive full credit if someone with no understanding of probability could simplify your answer to obtain the correct numerical solution.

A true/false question is true only if it is always true. If any conditions exist for which the statement is not true, then mark "false".

If you need more than sixty minutes, then submit your exam at the end of the test period, talk to nobody, and walk with the instructor to Grissom Hall, where you will given more time.

Score ______________________________

Closed book and notes. 60 minutes.

1. True or false. (For each: two points if correct, zero points if blank or wrong)

   (a) T ← F If events A and B are independent, then $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$.

   (b) T ← F If A and B are events, then $\mathrm{P(A)} = \mathrm{P}(A \mid B)\,\mathrm{P}(B) + \mathrm{P}(A \mid B')\,\mathrm{P}(B')$.

   (c) T F ← If X is a binomial random variable, then $\mathrm{F}_X(0) = 0$.

   (d) T F ← If A and B are events, then $\mathrm{P}(A \mid B) = \mathrm{P(B)}$ guarantees than A and B are independent.

   (e) T F ← If $f_X(c) = (2c+1)/25$ for $c = 0, 1, 2, 3, 4$ and zero elsewhere, then $F_X(1.5) = 0$.

   (f) T F ← If $V(X) = 0$, then $\mathrm{P(X = 0)} = 1$.

   (g) T F ← If $\mathrm{P}(E_1) + \mathrm{P}(E_2) + \mathrm{P}(E_3) = 1$, then the events E_1, E_2, and E_3 partition the sample space.

   (h) T F ← If Y_1, Y_2, and Y_3 are independent geometric random variables with probability of success (on each trial) q, then $Y = Y_1 + Y_2 + Y_3$ is negative binomial with parameter values $p = 3\,q$ and $r = 3$.

   (i) T ← F If f_X is a probability mass function, then $\sum_{\text{all } x} f_X(x) = 1$.

2. (Montgomery and Runger, Edition 3, Problem 3–70). A particularly long traffic light on your morning commute is green 20% of the time that you approach it. Assume that each morning represents a Bernoulli trial. Consider a random week, composed of five commuting days. Let X denote the number of days in which the light is not green. (Assume one commute per day.)

   (a) (4 points) X has a binomial distribution. What are its parameter values?

   $n = 5$
   $p = 0.8$

   (b) (5 points) What is the mean of X? (include the units)

   $\mathrm{E}(X) = n\,p = (5)\,(0.8) = 4$ days.

   (c) (7 points) What is the probability that the light is green on exactly one day (of the five)?

   $\mathrm{P}(X = 4) = C_4^5\; p^4\,(1-p)^{5-4} = (5)\,(0.8)^4\,(1-0.8) = (0.8)^4 = (0.62)^2 = 0.4096$ ←

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3. (one point each) Consider a sequence of Bernoulli trials. Circle the properties that are true.

   (a) ← All trials are independent.

   (b) All trials are mutually exclusive.

   (c) ← Every trial has exactly two possible outcomes.

   (d) ← All trials have the same probability of "success".

   (e) The probabilities of success sum to one.

4. Suppose that X is 1 with probability 0.4 and is 0 with probability 0.6.

   (a) (6 points) Write the pmf of X.

$f_X(0) = 0.6$
$f_X(1) = 0.4$
$f_X(c) = 0$ otherwise.
or
$f_X(c) = 0.6$ if $c = 0$
$f_X(c) = 0.4$ if $c = 1$
$f_X(c) = 0$ otherwise.
or
$f_X(c) = (0.4)^c (0.6)^{1-c}$ if $c = 0, 1$
$f_X(c) = 0$ otherwise.

   (b) (6 points) Determine the value of $E(X)$.

$$E(X) = \sum_{\text{all } c} c\ f_X(c) = (0)(0.6) + (1)(0.4) = 0.4 \leftarrow$$

   (c) (6 points) Determine the value of $E(X^2)$.

$$E(X^2) = \sum_{\text{all } c} c^2 f_X(c) = (0)^2(0.6) + (1)^2(0.4) = 0.4 \leftarrow$$

5. (Montgomery and Runger, Edition 3, Problem 3–75). Assume that each of your calls to a popular radio station has a probability of 0.04 of connecting (that is, not getting a busy signal). Assume that whether you get a busy signal on a particular call is independent of whether you get a busy signal on another call. Let Y denote the number of the call on which you get the first busy signal. That is, if your first attempt to connect yields a busy signal, then $Y = 1$.

(a) (3 points) What is the family of distributions of Y.

geometric $\leftarrow$

(b) (4 points) What are the values of the parameter(s)?

"success" is a busy signal, so $p = 1 - 0.4 = 0.96 \leftarrow$

(c) (5 points) What is the expected number of calls until you get a busy signal?

$\mathrm{E}(X) = 1/p = 1/0.96 \approx 1.04 \leftarrow$

(d) (6 points) What is the value of $\mathrm{P}(Y \geq 5)$?

$\mathrm{P}(Y \geq 5) = 1 - \mathrm{P}(Y < 5) = 1 - [\mathrm{P}(Y = 1) + \mathrm{P}(Y = 2) + \mathrm{P}(Y = 3) + \mathrm{P}(Y = 4)$
where $\mathrm{P}(Y = c) = p\,(1-p)^{c-1}$.
or
$\mathrm{P}(Y \geq 5) = \mathrm{P}(Y > 4) = \mathrm{P}(\text{first four calls connected}) = (0.04)^4 = 0.0000256 \leftarrow$

6. (Montgomery and Runger, Edition 3, Problem 2–107). Consider updating a computer database. There are two types of updates: addition of new records and changes to existing records. Adding a record takes less than 100 milliseconds, ninety percent of the time. Changing a record takes less than 100 milliseconds, twenty percent of the time. Thirty percent of requests are for adding new records.

(a) (4 points) Define event notation for this problem (so that you can answer Parts (b–d). (No probabilities here.)

Let A denote the event that the update request is to add a record.
Let F denote the event that the update request is fast (less than 100 milliseconds).
(Then A' denotes the event that the update request is to change a record.)

(b) (4 points) Write the given probability information using your event notation from Part (a).

$\mathrm{P}(\mathrm{F} \mid A) = 0.9$
$\mathrm{P}(\mathrm{F} \mid A') = 0.2$
$\mathrm{P}(A) = 0.3$

(c) (4 points) In your notation, write the (long-run) fraction of requests that are processed in less than 100 milliseconds. (No calculation here.)

$\mathrm{P}(F) \leftarrow$

(d) (8 points) Find the value for the fraction in Part (c).

$$\begin{aligned}
\mathrm{P}(F) &= \mathrm{P}(F \mid A)\,\mathrm{P}(A) + \mathrm{P}(F \mid A')\,\mathrm{P}(A') && \text{total probability}\\
&= (0.9)(0.3) + (0.2)(1 - 0.3) && \text{Part (b)}\\
&= (0.27) + (0.14) && \text{simplify}\\
&= 0.41 \leftarrow && \text{simplify}
\end{aligned}$$

Discrete Distributions: Summary Table

random variable	distribution name	range	probability mass function	expected value	variance
X	general	$x_1, x_2, \ldots, x_n$	$P(X = x)$ $= f(x)$ $= f_X(x)$	$\sum_{i=1}^{n} x_i f(x_i)$ $= \mu = \mu_X$ $= E(X)$	$\sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$ $= \sigma^2 = \sigma_X^2$ $= V(X)$ $= E(X^2) - \mu^2$
X	discrete uniform	$x_1, x_2, \ldots, x_n$	$1/n$	$\sum_{i=1}^{n} x_i / n$	$[\sum_{i=1}^{n} x_i^2 / n] - \mu^2$
X	equal-space uniform	$x = a, a+c, \ldots, b$ where	$1/n$ $n = (b-a+c)/c$	$\frac{a+b}{2}$	$\frac{c^2(n^2-1)}{12}$
"# successes in 1 Bernoulli trial"	indicator variable	$x = 0, 1$	$p^x (1-p)^{1-x}$	p where	$p(1-p)$ $p = P(\text{"success"})$
"# successes in n Bernoulli trials"	binomial	$x = 0, 1, \ldots, n$	$C_x^n p^x (1-p)^{n-x}$	np where	$np(1-p)$ $p = P(\text{"success"})$
"# successes in a sample of size n from a population of size N containing K successes"	hyper-geometric (sampling without replacement)	$x = (n-(N-K))^+, \ldots, \min\{K, n\}$ and integer	$C_x^K C_{n-x}^{N-K} / C_n^N$	np where	$np(1-p)\frac{(N-n)}{(N-1)}$ $p = K/N$
"# Bernoulli trials until 1st success"	geometric	$x = 1, 2, \ldots$	$p(1-p)^{x-1}$	$1/p$ where	$(1-p)/p^2$ $p = P(\text{"success"})$
"# Bernoulli trials until rth success"	negative binomial	$x = r, r+1, \ldots$	$C_{r-1}^{x-1} p^r (1-p)^{x-r}$	r/p where	$r(1-p)/p^2$ $p = P(\text{"success"})$
"# of counts in time t from a Poisson process with rate λ"	Poisson	$x = 0, 1, \ldots$	$e^{-\mu} \mu^x / x!$	μ where	μ $\mu = \lambda t$

Result. For $x = 1, 2, \ldots$, the geometric cdf is $F_X(x) = 1 - (1-p)^x$.

Result. The geometric distribution is the only discrete memoryless distribution.
That is, $P(X > x + c \mid X > x) = P(X > c)$.

Result. The binomial distribution with $p = K / N$ is a good approximation to the hypergeometric distribution when n is small compared to N.