IE 230 Seat # ________ Name (neatness, 1 point) ___ < KEY > ___

Closed book and notes. 60 minutes.

Cover page and four pages of exam.
No calculators.

This test covers event probability, Chapter 2 of Montgomery and Runger, fourth edition.

Score ____________________________

Closed book and notes. 60 minutes.

1. True or false. (3 points each)

   (a) T ← F If A is an event, then A and A' are mutually exclusive.

   (b) T ← F If A is an event, then A and A' partition the sample space.

   (c) T F ← If $\mathrm{P}(A \mid B) = 0.5$, $\mathrm{P}(B) = 0.5$, and $\mathrm{P}(A) = 0.7$, then B' and A' are independent.

   (d) T F ← If A and B are independent events, then $\mathrm{P}(B \mid A) = \mathrm{P}(A)$.

   (e) T ← F If $\varnothing$ denotes the empty set, then $\mathrm{P}(\varnothing) = 0$.

   (f) T F ← If A and B are events, then $\mathrm{P}(A \mid B) \geq \mathrm{P}(A)$.

   (g) T F ← If A is an event, then $\mathrm{P}(A \mid A \cup A') = 0$.

   (h) T ← F If A and B are positive-probability events, then $\mathrm{P}(A \mid B)\,\mathrm{P}(B) = \mathrm{P}(B \mid A)\,\mathrm{P}(A)$.

   (i) T F ← If A and B are events and A is a subset of B, then $\mathrm{P}(B) \leq \mathrm{P}(A)$.

2. Recall: A 52-card deck has four suits (spades, hearts, diamonds, and clubs) and thirteen values (Ace, 2, 3,...,10, Jack, Queen, and King). Spades and clubs are black; hearts and diamonds are red.

   Consider the experiment of choosing ten cards without replacement. For draw $i = 1, 2, \ldots, 10$, let Q_i denote that a queen is chosen, let K_i denote that a king is chosen, and let A_i denote that an ace is chosen.

   (a) (3 points) T F ← $Q_1 \mid K_9$ is an event.

   (b) (3 points) T ← F Q_1 and K_1 are mutually exclusive

   (c) (3 points) T F ← $\mathrm{P}(Q_9 \cap K_9) = 8/52$.

   (d) (7 points) Determine the value of $\mathrm{P}(Q_1 \mid A_9 \cap K_8)$.

   $4/(52-2)$ ←

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3. (from Montgomery and Runger, 2–147) A steel plate contains 30 bolts. Assume that 7 bolts are not torqued to the proper limit. Five bolts are selected at random, without replacement, to be checked for torque.

(a) (7 points) Determine the probability that all five of the selected bolts are torqued to the proper limit.

For $i = 1, 2, ..., 5$, let B_i denote the event that the ith bolt is torqued properly.

P(all five torqued properly)

$$\begin{aligned} &= \mathrm{P}(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5) \\ &= \mathrm{P}(B_1)\,\mathrm{P}(B_2 \mid B_1) \cdots \mathrm{P}(B_5 \mid B_1 \cap B_2 \cap \cdots \cap B_4) \\ &= (23/30)\,(22/29)\,(21/28)\,(20/27)\,(19/26) \leftarrow \end{aligned}$$

(b) (7 points) Determine the probability that at least one of of the selected bolts is not torqued to the proper limit.

For $i = 1, 2, ..., 5$, let B_i denote the event that the ith bolt is torqued properly.

P(at least one torqued improperly)

$$\begin{aligned} &= 1 - \mathrm{P}(\text{all five torqued properly}) \\ &= 1 - \mathrm{P}(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5) \\ &= 1 - \mathrm{P}(B_1)\,\mathrm{P}(B_2 \mid B_1) \cdots \mathrm{P}(B_5 \mid B_1 \cap B_2 \cap B_3 \cap B_4) \\ &= 1 - (23/30)\,(22/29)\,(21/28)\,(20/27)\,(19/26) \leftarrow \end{aligned}$$

(c) (7 points) Determine the probability that the third bolt is torqued to the proper limit.

$(30 - 7)/30 \leftarrow$

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4. Result: For any events A and B, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.

(3 points each) For each of the six lines, state why the corresponding equality is true.

Each blank requires one reason; reasons may be reused. Choose the reasons from this list:

(i) Events partition the sample space.

(ii) Events are complementary.

(iii) Events are mutually exclusive.

(iv) Events are independent.

(v) Events are the same.

(vi) Definition of conditional probability.

(vii) Multiplication Rule.

(viii) Total Probability.

(ix) Bayes's Rule.

(x) Algebra (i.e., no set theory or probability needed).

$\mathrm{P}(A \cup B) = \mathrm{P}[(A \cap B') \cup (A \cap B) \cup (A' \cap B)]$ (a) ___ < v > ___

$= \mathrm{P}(A \cap B') + \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B)$ (b) ___ < iii > ___

$= [\mathrm{P}(A \cap B') + \mathrm{P}(A \cap B)] + [\mathrm{P}(A' \cap B) + \mathrm{P}(A \cap B)]$
$+ [\mathrm{P}(A \cap B) - 2\,\mathrm{P}(A \cap B)]$ (c) ___ < x > ___

$= [\mathrm{P}(A \cap B') + \mathrm{P}(A \cap B)] + [\mathrm{P}(A' \cap B) + \mathrm{P}(A \cap B)]$
$- \mathrm{P}(A \cap B)$ (d) ___ < x > ___

$= \mathrm{P}[(A \cap B') \cup (A \cap B)] + \mathrm{P}[(A' \cap B) \cup (A \cap B)]$
$- \mathrm{P}(A \cap B)$ (e) ___ < iii > ___

$= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ (f) ___ < v > ___

5. A system is composed of three components, each of which randomly works or doesn't work. The system works if at least one component works. Let A denote the event that the first component works, B the second component works, and C the third component works.

The first component works one-third of the time. The second component works one tenth of the time, independently of the first component. If both of the first two components work, then the third component works one time in ten; otherwise, the third component works one-seventh of the time.

(a) (4 points) Write the four given probabilities in the defined notation.

$\mathrm{P}(A) = 1/3 \leftarrow$

$\mathrm{P}(B) = \mathrm{P}(B \mid A) = 1/10 \leftarrow$

$\mathrm{P}(C \mid A \cap B) = 1/10 \leftarrow$

$\mathrm{P}(C \mid (A \cap B)') = 1/7 \leftarrow$

(b) (3 points) T F $\leftarrow$ The events A and B are mutually exclusive.

(c) (3 points) T $\leftarrow$ F The event that the system works can be written as $(A' \cap B' \cap C')'$.

(d) (7 points) Determine the probability that the system works.

$$\begin{aligned}
\mathrm{P}[(A' \cap B' \cap C')'] &= 1 - \mathrm{P}(A' \cap B' \cap C') \\
&= 1 - \mathrm{P}(A')\,\mathrm{P}(B' \mid A')\,\mathrm{P}(C' \mid A' \cap B') \\
&= 1 - \mathrm{P}(A')\,\mathrm{P}(B')\,\mathrm{P}(C' \mid A' \cap B') \\
&= 1 - [1 - \mathrm{P}(A)]\,[1 - \mathrm{P}(B)]\,[1 - \mathrm{P}(C \mid A' \cap B')] \\
&= 1 - (2/3)\,(9/10)\,(6/7) \leftarrow
\end{aligned}$$
