IE 230   Seat # ____________   (1 point) Name (clearly) ___ < KEY > ___

Closed book and notes. No calculators.

Designed for 60 minutes, but time is essentially unlimited.

Cover page, four pages of exam.

This test covers through Section 2.7 of Montgomery and Runger, fourth edition.

Throughout, you will receive full credit if someone with no understanding of probability could simplify your answer to obtain the correct numerical solution.

A true/false question is true only if it is always true. If any conditions exist for which the statement is not true, then mark "false".

Score ______________________________

Closed book and notes. 60 minutes.

1. True or false. (For each: three points if correct, zero point if blank, −1 point if wrong)

(a) T ← F If events A and B partition the sample space, then $P(A) + P(B) = 1$.

(b) T F ← If events A and B are mutually exclusive, then $P(A) + P(B) = 1$.

(c) T F ← Bayes's Rule applies only to independent events.

(d) T F ← For all events A and B, $P(A \mid B) = P(A \cap B) / P(B)$.

(e) T ← F For all events A and B, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.

(f) T ← F For all events A and B, $P(B) = P(B \mid A)\,P(A) + P(B \mid A')\,P(A')$.

(g) T ← F For all independent events A and B, $P(A \cap B) = P(A)\,P(B)$.

(h) T ← F If event A is a subset of the event B, then $P(A) \le P(B)$.

(i) T ← F If n and r are non-negative integers with $r \le n$, then

$$C_r^n = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}\,.$$

2. (10 points) (Montgomery and Runger, Edition 4, Problem 2–47). In a chemical plant, twenty-four holding tanks are used for final product storage. Four tanks are selected at random and without replacement. Six of the twenty-four tanks contains material in which the viscosity exceeds the customer requirements.

Determine the probability that at least one tank in the sample (of four tanks) contains high-viscosity material.

Experiment: Randomly choose, without replacement, four tanks from the 24 tanks.

Let H_i = "the ith tank chosen contains high-viscosity material", $i = 1, 2, 3, 4$.

Then P("at least one of the four tanks contains high-viscosity material")

= 1 − P("none of the four tanks contains high-viscosity material")

$= 1 - P(H_1' \cap H_2' \cap H_3' \cap H_4')$

$= 1 - P(H_1')\,P(H_2' \mid H_1')\,P(H_3' \mid H_1' \cap H_2')\,P(H_4' \mid H_1' \cap H_2' \cap H_3')$

$= 1 - [(18/24)\,(17/23)\,(16/22)\,(15/21)]$ ←

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3. (ten points) (Montgomery and Runger, Edition 4, Problem 2–88). Suppose that A and B are mutually exclusive events. Construct a Venn diagram that contains the three events A, B, and C such that $\mathrm{P}(A \mid C) = 0$ and $\mathrm{P}(B \mid C) = 1$.

To represent the sample space, sketch a rectangle and label it S.

To represent mutually exclusive events, sketch two non-overlapping circles in the rectangle; label them A and B.

To represent the two probability conditions, sketch a circle inside circle B; label it C.

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4. (Montgomery and Runger, Edition 4, Problem 2–117). Suppose that $\mathrm{P}(A \mid B) = 0.4$, $\mathrm{P}(A \mid B') = 0.2$, and $\mathrm{P}(B) = 0.8$.

(a) (six points) Determine the value of $\mathrm{P}(A)$.

$\mathrm{P}(A)$	$= \mathrm{P}(A \mid B)\,\mathrm{P}(B) + \mathrm{P}(A \mid B')\,\mathrm{P}(B')$	total probability
	$= \mathrm{P}(A \mid B)\,\mathrm{P}(B) + \mathrm{P}(A \mid B')\,[1 - \mathrm{P}(B)]$	complement ←
	$= (0.4)\,(0.8) + (0.2)\,[1 - 0.8]$	given information
	$= 0.36$	simplify

(b) (six points) Determine the value of $\mathrm{P}(B \mid A)$.

$\mathrm{P}(B \mid A)$	$= \mathrm{P}(A \cap B) / \mathrm{P}(A)$	definition of conditional probability
	$= \mathrm{P}(A \mid B)\,\mathrm{P}(B) / \mathrm{P}(A)$	multiplication rule ←
	$= (0.4)\,(0.8) / 0.36$	given information
	$= 0.32 / 0.36$	given information

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5. (Montgomery and Runger, Edition 4, Problem 2–115). The following circuit operates if and only if there is a path of functional devices from left to right. The probability that each devices functions is shown. Assume that the probability that a device functions is independent of the other devices. We are interested in the probability that the circuit operates?

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                          -------
                       --| 0.90 |--
        --------      |   ------     |    ------
     ---| 0.97  |--                    --| 0.70 |---
        --------      |   ------     |    ------
                       --| 0.15 |--
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(a) (four points) What is the experiment?

Randomly choose a circuit. $\leftarrow$

(Maybe different assemblies or maybe one circuit at a random time.)

(b) (five points) Define notation and write the given information in your notation.

Let D_i = "device i functions, $i = 1, 2, 3, 4$.

Then $\mathrm{P}(D_1) = 0.97$

$\mathrm{P}(D_2) = 0.90$

$\mathrm{P}(D_3) = 0.15$

$\mathrm{P}(D_4) = 0.70$

and D_1, D_2, D_3 and D_4 are independent.

(c) (five points) In your notation, write the event that the circuit operates.

$D_1 \cap (D_2 \cup D_3) \cap D_4$ or $(D_1 \cap D_2 \cap D_4) \cup (D_1 \cap D_3 \cap D_4)$

(d) (five points) Find the probability of the event in Part (c).

$\mathrm{P}(D_1 \cap (D_2 \cup D_3) \cap D_4)$

$= \mathrm{P}(D_1)\, \mathrm{P}(D_2 \cup D_3) \cap D_4)$ independent events

$= \mathrm{P}(D_1)\, [\mathrm{P}(D_2) + \mathrm{P}(D_3) - \mathrm{P}(D_2 \cap D_3)]\, \mathrm{P}(D_4)$ expand the union

$= \mathrm{P}(D_1)\, [\mathrm{P}(D_2) + \mathrm{P}(D_3) - \mathrm{P}(D_2)\, \mathrm{P}(D_3)]\, \mathrm{P}(D_4)$ independence $\leftarrow$

$= (0.97)\, [(0.90) + (0.15) - (0.90)\, (0.15)]\, (0.70)$ given

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6. (Montgomery and Runger, Edition 4, Problem 2–135). A congested computer network has a 0.004 probability of losing a data packet. Packet loses are independent. A lost packet must be resent.

(a) (five points) What is the probability that an email message with 100 packets will need no packet to be resent?

Experiment: Choose a random message with 100 packets.
Let R_i = "the i th packet must be resent" for $i = 1, 2, ..., 100$.
Then for every i, $\text{P}(R_i) = 0.004$ and, by implication, $\text{P}(R_i{}') = 0.996$.

P("no packet is resent")	$= \text{P}(R_1{}' \cap R_2{}' \cap \cdots \cap R_{100}{}')$	same event
	$= \text{P}(R_1{}')\,\text{P}(R_2{}') \cdots \text{P}(R_{100}{}')$	independence ←
	$= (0.996)^{100}$	given information

(b) (six points) What is the probability that an email message with four packets will need exactly one packet to be resent?

Experiment: Choose a random message with 4 packets.
Let R_i = "the i th packet must be resent" for $i = 1, 2, 3, 4$.
Then for every i, $\text{P}(R_i) = 0.004$ and, by implication, $\text{P}(R_i{}') = 0.996$.
The event "exactly one packet is resent" can be written as

$$[R_1 \cap R_2{}' \cap R_3{}' \cap R_4{}'] \cup [R_1{}' \cap R_2 \cap R_3{}' \cap R_4{}'] \cup [R_1{}' \cap R_2{}' \cap R_3 \cap R_4{}'] \cup [R_1{}' \cap R_2{}' \cap R_3{}' \cap R_4]$$

The four events in brackets are mutually exclusive, so sum their probabilities.
The four probabilities are equal, so consider only the first one:
Independence yields $\text{P}(R_1 \cap R_2{}' \cap R_3{}' \cap R_4{}') = \text{P}(R_1)\,(\text{R}_2{}')\,(\text{R}_3{}')\,(\text{R}_4{}')$.
The given information yields $(0.004)\,(0.996)^3$.
Therefore, P("exactly one packet is resent") $= 4\,(0.004)\,(0.996)^3$ ←

(c) (six points) Suppose that ten email messages are sent, each with 100 packets. Determine the probability that at least one message will need at least one packet to be resent?

Experiment: Choose a random message with 100 packets.
Let M_i = "the i th message has at least one packet resent" for $i = 1, 2, ..., 10$.
Then, from Part (a), $\text{P}(M_i{}') = (0.996)^{100}$ for $i = 1, 2, ..., 10$.
P("at least one message will need at least one packet resent")

$= 1 - $ P("no message needs at least one packet resent")	complement
$= 1 - \text{P}(M_1{}' \cap M_2{}' \cap \cdots \cap M_{10}{}')$	same event
$= 1 - \text{P}(M_1{}')\,\text{P}(M_2{}') \cdots \text{P}(M_{10}{}')$	independence ←
$= 1 - [(0.996)^{100}]^{10}$	given information
$= 1 - (0.996)^{1000}$	simplify

Alternatively (and easier), notice that all 1000 packets must not be resent.