**Math 265 Linear Algebra** **Sample** **Spring 2000**

**1.** It is given that $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$ and $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

**(a)** Find the rank of $A$.

**(b)** Find the nullity of $A$.

**(c)** Find a basis for the column space of $A$.

**(d)** Find a basis for the row space of $A$.

**(e)** Find a basis for the null space of $A$.

**(f)** Find a basis for the orthogonal complement of the row space of $A$.

**2.** It is given that $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 1 & -4 \\ -1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 3 & 6 & 2 & -5 \end{bmatrix}$ and $\mathbf{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

**(a)** Find the rank of $A$.

**(b)** Find the nullity of $A$.

**(c)** Find a basis for the column space of $A$.

**(d)** Find a basis for the row space of $A$.

**(e)** Find a basis for the null space of $A$.

**(f)** Find a basis for the orthogonal complement of the row space of $A$.

**3.** Find an equation relating $a, b$ and $c$ so that the linear system

$$\begin{cases} 2x + 2y + 3z = a \\ 3x - y + 5z = b \\ x - 3y + 2z = c \end{cases}$$

is consistent for any values $a, b$ and $c$ which satisfy that equation.

**4.** Determine the values of $a$ so that the linear system

$$\begin{cases} x + y + z = 2 \\ 2x + 3y + z = 5 \\ 2x + 3y + (a^2 - 1)z = a + 1 \end{cases}$$

has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

**5.** Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}.$$

Find $L\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)$.

**6.** Let $L:\mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix}-2\\1\\-2\end{bmatrix}\right)=\begin{bmatrix}-3\\1\\2\end{bmatrix},L\left(\begin{bmatrix}3\\2\\-1\end{bmatrix}\right)=\begin{bmatrix}2\\-2\\1\end{bmatrix},L\left(\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\2\\4\end{bmatrix}.$$

Find $L\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)$.

**7.** Find the standard matrix of the linear transformation $L$ defined by

$$L\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}2x-z\\x+2y+z\\3x-y\end{bmatrix}.$$

**8.** Find an equation for the plane through the point $(1,-3,-2)$ parallel to the plane $3x+4y-6z=5$.

**9.** Find an equation of the plane containing the two lines $\dfrac{x-3}{2}=\dfrac{y+1}{-3}=\dfrac{z-2}{4}$ and $\dfrac{x-3}{2}=\dfrac{y+1}{1}=\dfrac{z-2}{-1}$.

**10.** Does the vector $(6,6,0,-1)$ belong to the span of $(1,1,0,1),(-1,1,2,0)$ and $(2,3,1,-1)$ ? If it does, write it as a linear combination of these three vectors.

**11.** For what values of $d$ are the vectors $(1,3,d),(1,1,0)$ and $(0,1,1)$ linearly independent ?

**12.** Compute the inverse of the matrix $A=\begin{bmatrix}2&0&1\\-2&1&3\\2&-1&1\end{bmatrix}$.

**13.** Compute the inverse of the matrix $A=\begin{bmatrix}2&-2&1\\-3&2&0\\4&1&1\end{bmatrix}$.

**14.** Find ALL diagonal matrices similar to $A=\begin{bmatrix}1&1\\-5&7\end{bmatrix}$.

**15.** Find a diagonal matrix similar to $A=\begin{bmatrix}0&4&4\\4&0&4\\4&4&0\end{bmatrix}$.

**16.** Find the cosine of the acute angle between the lines

$$l_1:\begin{cases}x=1+2t\\y=-2-t\\z=4-2t\end{cases}\quad l_2:\begin{cases}x=1+2s\\y=-2-s\\z=4+s\end{cases}$$

**17.** Use the Gram-Schmidt process to construct an orthonormal basis for the subspace $W$ of $\mathbb{R}^4$ spanned by

$$(1,1,0,0),(2,-1,0,1),(3,-3,0,-2),(1,-2,0,-3).$$

**18.** Let $W$ be the subspace of $\mathbb{R}^3$ spanned by the vector $w = (1, 1, -1)$.

**(a)** Find a basis for the orthogonal complement $W^\perp$ of $W$.

**(b)** Find an orthonormal basis for $W^\perp$.

**19.** Let $W$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 2, 0, -2)$, $(0, 1, 2, 1)$ and $(4, -1, 0, 1)$. Find the projection of $(1, -1, 0, 4)$ onto $W$.

**20.** Determine if each of the following statements is true for *every* $5 \times 7$ matrix $A$ with $\text{rank}(A) = 3$.

**(a)** For every $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is uniquely solvable. True False

**(b)** For some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is uniquely solvable, and for some $\mathbf{b}$, it is not solvable. True False

**(c)** For every $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. True False

**(d)** For some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, and for some $\mathbf{b}$, it is not solvable. True False

**(e)** For some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is not solvable. True False

**21.** $A$, $B$ and $C$ are $3 \times 3$ matrices and $k$ is a scalar. Determine if each of the following statements is *always* true.

**(a)** If $A^2 = I_3$, then $A = I_3$ or $-I_3$. True False

**(b)** $\det(AB) = \det(A)\det(B)$. True False

**(c)** $\det(kA) = k^3 \det(A)$. True False

**(d)** If $C$ is invertible, the characteristic polynomial of $A$ is the same as that of $C^{-1}AC$. True False

**(e)** $\det(\text{adj(A)}) = \det(\text{A})$. True False

**22.** For each of the following sets of vectors, determine if it is a vector (sub)space:

**(a)** The set of all vectors in $\mathbb{R}^4$ with the property $2x_1 + x_2 - 3x_3 + x_4 = 0$; Yes No

**(b)** The set of all vectors in $\mathbb{R}^4$ with the property $x_1^3 = x_2^3, x_3 = x_4$; Yes No

**(c)** The set of all vectors in $\mathbb{R}^3$, which have the form $(0, a - b + c, 3b + c)$ where $a, b$ and $c$ are arbitrary real numbers; Yes No

**(d)** The set of all polynomials $P$ in the space of all polynomials of degree at most 7, with the property $P(2) = 0$; Yes No

**(e)** The set of all nonsingular matrices in the space of $3 \times 3$ matrices; Yes No

**23.** Determine if each of the following sets of vectors is linearly independent or linearly dependent:

**(a)** $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \right\}$; Independent Dependent

**(b)** $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$; Independent Dependent

**(c)** $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \right\}$; Independent Dependent

**(d)** $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} \right\}$; Independent Dependent

**(e)** $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Independent Dependent

**24.** We have a $3 \times 3$ matrix $A = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ with $\det(A) = 3$. Compute the determinant of the following matrix

**(a)** $\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$

**(b)** $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

**(c)** $\begin{bmatrix} a & 1 & 1 \\ b & 3 & 3 \\ c & 5 & 5 \end{bmatrix}$

**(d)** $A^T$

**(e)** $(5A)^{-1}$

**25.** Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}.$$

Determine if each of the following statement is true or false.

**(a)** If any two of $a, b, c$ have the same value, then $\det(A) = 0$. True False

**(b)** If $a, b, c$ have distinct values, then $A$ is nonsingular. True False

**(c)** If $a > b > c$, then $\det(A) < 0$. True False

**(d)** If $\text{rank(A)} = 1$, then $a, b$ and $c$ must all be equal. True False

**(e)** The rank of $A$ can never be equal to 2. True False

**26.** Let $A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$.

**(a)** Find all eigenvalues of $A$.

**(b)** For each eigenvalue above, find an eigenvector of $A$ associated to it.

**(c)** Find a diagonal matrix $D$ and a nonsingular matrix $P$ such that $P^{-1}AP = D$.

**(d)** Find $A^{47}$.

**27.** Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

**(a)** Find all eigenvalues of $A$.

**(b)** For each eigenvalue as above, find an eigenvector of $A$ associated to it.

**(c)** Find a diagonal matrix $D$ and an orthogonal matrix $P$ such that $P^T AP = D$.

**28.** Determine an invertible matrix $\hat{A}$ and a vector $\hat{b}$ such that the solution to $\hat{A}\hat{X} = \hat{b}$ is the least squares solution to $AX = b$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

**29.** Find the least squares fit line for the points

$$(1, 2), (2, 1), (3, 3), (4, 3).$$

**30.** We want to find the least squares quadratic polynomial for the points

$$(1, 1), (3, 0), (4, 3), (6, 4).$$

**(a)** If $y = b_2x^2 + b_1x + b_0$ were a parabola passing through all the points, then $X = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$ would satisfy the equation $AX = \mathbf{b}$ for a matrix $A$ and a vector $\mathbf{b}$. Write down $A$ and $\mathbf{b}$.

**(b)** If $y = \hat{b}_2x^2 + \hat{b}_1x + \hat{b}_0$ is the least sqaures quadratic polynomial, then $\hat{X} = \begin{bmatrix} \hat{b}_2 \\ \hat{b}_1 \\ \hat{b}_0 \end{bmatrix}$ satisfies the equation $\hat{A}\hat{X} = \hat{\mathbf{b}}$ for a matrix $\hat{A}$ and a vector $\hat{\mathbf{b}}$. Write down $\hat{A}$ and $\hat{\mathbf{b}}$.

**(c)** Find the least squares quadratic polynomial.

**31.** Let

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the linear system of differential equations where

$$A = \begin{bmatrix} 2 & -1 \\ 8 & 8 \end{bmatrix}.$$

**(a)** Find all eigenvalues of $A$.

**(b)** Find the general solution to the linear system of differential equations.

**(c)** Find the solution given the initial condition of

$$x_1(0) = 3, x_2(0) = -10.$$

**32.** Let

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the linear system of differential equations where

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

**(a)** Find all eigenvalues of $A$.

**(b)** Find the REAL general solution to the linear system of differential equations.

**(c)** Find the solution given the initial condition of

$$x_1(0) = 5, x_2(0) = 1.$$

**33.** Let

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be the linear system of differential equations where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

**(a)** Find all eigenvalues of $A$.

**(b)** Find the general solution to the linear system of differential equations.

**(c)** Find the solution given the initial condition of

$$x_1(0) = -2, x_2(0) = 1, x_3(0) = 2.$$