Math 265 Linear Algebra — Final Exam — Spring 2000

Student Name (print):	Student ID:

Circle the name of your instructor (with the time of your class):

Archava	Ban (10:30)	Ban (11:30)	Brown
Eremenko	Feng	Heinzer	Matsuki
Thurber (7:30)	Thurber (11:30)	Pang (12:30)	Pang (1:30)

Do not write below this line.

Please be neat and show all work. Write each answer in the provided box. Use the back of the sheets and the last 3 pages for extra scratch space. Return this entire booklet to your instructor. **No books. No notes. No calculators.**

Problem #	Max pts.	Earned points
1	20	
2	7	
3	7	
4	7	
5	7	
6	7	
7	7	
8	7	
9	7	
10	7	
11	12	
Section I	**95**	

12	5	
13	10	
14	10	
15	10	
16	10	
Section II	**45**	
17	20	
18	20	
19	20	
Section III	**60**	
TOTAL	**200**	

Section I: Short problems

No partial credit on this part, but show all your work anyway. It might help you if you come close to a borderline. Please be neat. Write your answer in the provided box.

1. It is given that $A = \begin{bmatrix} 1 & 2 & -1 & 0 & -3 \\ 2 & 4 & -2 & 1 & -4 \\ -1 & -2 & 1 & 0 & 3 \end{bmatrix}$ and $\mathbf{rref}(A) = \begin{bmatrix} 1 & 2 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find the rank of A.

(b) Find the nullity of A.

(c) Find a basis for the column space of A.

(d) Find a basis for the row space of A.

(e) Find a basis for the null space of A.

2. Find an equation relating a, b and c so that the linear system

$$\begin{cases} x + 2y - 3z = a \\ 2x + 3y + 3z = b \\ 5x + 9y - 6z = c \end{cases}$$

is consistent for exactly the values a, b and c which satisfy that equation.

3. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\-1\\1\end{bmatrix}, L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\2\end{bmatrix}, L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\4\\2\end{bmatrix}.$$

Find $L\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$.

4. Find an equation for the plane through the point $(1, -3, -2)$ perpendicular to the line $\dfrac{x-2}{3} = \dfrac{y+3}{-1} = \dfrac{z-4}{2}$.

5. Find all the values for k such that the vectors

$$(1, 3, k, 1), (1, 1, 0, -1), (0, 1, 1, 1)$$

are linearly dependent.

6. Compute the inverse of the matrix $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$.

7. Find a diagonal matrix similar to $A = \begin{bmatrix} 4 & 1 \\ -3 & 8 \end{bmatrix}$.

8. Find all eigenvalues for $A = \begin{bmatrix} -2 & 3 & -1 \\ 0 & 2 & 3 \\ 0 & -1 & 6 \end{bmatrix}$

9. Find the cosine of the angle PQR (the angle at Q in the triangle with vertices P, Q, R) where

$$P = (4, 2, 3), Q = (1, -2, 3), R = (2, 0, 1).$$

10. Let W be the subspace of $\mathbb{R}^3$ spanned by $(1,0,1)$ and $(-1,2,1)$. Find the projection of the vector $(1,1,1)$ onto W.

11. Apply the Gram-Schmidt method to find an *orthonormal* basis for the span of the three vectors (in the given order): $(1,1,0,1), (0,1,1,1)$, and $(1,1,1,1)$.

Section II: Multiple choice problems

For Problem 12, choose only one true statement. For Problems 13 through 16, circle only one (the correct) answer for each part.
No partial credit.

12. Choose one statement from below which is true for $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -7 \end{bmatrix}$. It is given that rank(A) = 2.

(a) For every $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is uniquely solvable

(b) For some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is uniquely solvable, and for some $\mathbf{b}$, it is not solvable.

(c) For every $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(d) For some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, and for some $\mathbf{b}$, it is not solvable.

(e) For every $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is not solvable.

13. Determine if each of the following statements is *always* true, where A, B and C are 2×2 matrices and k is a scalar.

(a) If $A^2 = A$, then $A = I_n, -I_n$ or 0. True False

(b) $\det(A + B) = \det(A) + \det(B)$. True False

(c) $\det(kA) = k\det(A)$. True False

(d) If C is invertible, the characteristic polynomial of A is the same as that of $C^{-1}AC$. True False

(e) $A\ \mathrm{adj(A)} = \det(\mathrm{A})\ \mathrm{I}_2$. True False

14. For each of the following sets of vectors, determine if it is a vector (sub)space:

(a) The set of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with the property $x_1 + x_2 = 0$; Yes No

(b) The set of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with the property $x_1^2 = x_2^2$; Yes No

(c) The set of all vectors in $\mathbb{R}^3$, which have the form $(a+b, a-b, c)$ where a, b and c are arbitrary real numbers; Yes No

(d) The set of all solutions to the linear system of differential equations $\dfrac{dX}{dt} = AX$ where $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$; Yes No

(e) The set of all symmetric matrices in the space of 3×3 matrices; Yes No

15. Determine if each of the following sets of vectors is linearly independent or linearly dependent:

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$; Independent Dependent

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$; Independent Dependent

(c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$; Independent Dependent

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$; Independent Dependent

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \right\}$. Independent Dependent

16. Let $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$. In each case below, does the given matrix always have the same determinant as A ?

(a) $\begin{bmatrix} 3a & 3b & 3c \\ c & a & b \\ \frac{1}{3}b & \frac{1}{3}c & \frac{1}{3}a \end{bmatrix}$. Yes No

(b) $\begin{bmatrix} a-c & b-a & c-b \\ c & a & b \\ b & c & a \end{bmatrix}$. Yes No

(c) $\begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}$. Yes No

(d) $A^T = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$. Yes No

(e) $-A = \begin{bmatrix} -a & -b & -c \\ -c & -a & -b \\ -b & -c & -a \end{bmatrix}$. Yes No

Section III: Multi-Step problems

Show all work (no work - no credit!) and display computing steps. Write clearly.

17. Find the least squares fit line for the points

$$(-4, 3), (-3, 4), (2, 3), (5, 4).$$

18. Let

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the linear system of differential equations where

$$A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}.$$

(a) Find all eigenvalues of A.

(b) Find the REAL general solution to the linear system of differential equations.

19. Let

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be the linear system of differential equations where

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find the eigenvalues and find an eigenvector for each eigenvalue.

(b) Find the solution to the linear system of differential equations given the initial condition of

$$x_1(0) = -4, x_2(0) = -4, x_3(0) = 2.$$