MATH 265 FINAL EXAM

Name and ID:

Instructor: **Section or class time:**

Instructions: Calculators are not allowed. There are 8 problems in the first part worth 13 points each. There are 12 problems in the second part worth 8 points each. The total is 200 points.

1		
2		
3		
4		
5		
6		
7		
8		
9	x	
10	x	
11	x	
12	x	

Part I

You must show all work for problems in first part.

1. Let

$$A = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$$

   (a) Find conditions on a, b and c such that A is symmetric.

   (b) Find conditions on a, b and c such that A has an inverse.

   (c) Find conditions on a, b and c such that A satisfies $A^2 = 0$.

2. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 0 & t-7 & 0 & 6 \\ 0 & 2 & 2t-2 & -2 \\ 1 & -1 & -2 & 1 \end{array}\right]$$

(a) For what values of t will the system have no solutions?

(b) For what values of t will the system have infinitely many solutions?

3. Find a basis for the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

4. Let W be the span of $(1, 0, 2)$ and $(0, 1, 0)$ in $\mathbb{R}_3$ (which is given the standard inner product.)

   (a) Find a basis for $W^\perp$.

   (b) Write the vector $\mathbf{v} = (1, 2, 3)$ as $\mathbf{w} + \mathbf{u}$ with $\mathbf{w}$ in W and $\mathbf{u}$ in $W^\perp$.

5. Let

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \\ 5 & 1 & 3 \end{bmatrix}$$

(a) Compute the eigenvalues of A.

(b) Is A diagonalizable? (Justify your answer.)

6. Let

$$A = \begin{bmatrix} 4 & -3 \\ 5 & -4 \end{bmatrix}.$$

(a) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(b) Compute A^{12}.

7. Consider the system of differential equations:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

You are given that the matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

has two eigenvalues namely 2 and 1. The eigenspace of the eigenvalue 2 is spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The eigenspace of the the eigenvalue 1 is spanned by

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Find the general solution.

(b) Find the solution to the initial value problem determined by $x_1(0) = 1$, $x_2(0) = 2$ and $x_3(0) = 3$.

8. Let A and B be two $n \times n$ matrices. Suppose that B is similar to A which means that $B = P^{-1}AP$ for some nonsingular $n \times n$ matrix P. Show that

   (a) if $\det(A) \neq 0$, then $\det(B) \neq 0$;

   (b) if A^{-1} exists, then B^{-1} exists and is similar to A^{-1}.

Part II: (Multiple Choice)

Circle the correct answer.

1. Assume that x, y, z satisfy

$$\begin{array}{ccccccc} 2x & + & ay & + & 3z & = & a \\ 5x & + & by & - & z & = & b \\ x & + & cy & + & z & = & c \end{array}$$

where

$$\det \begin{bmatrix} 2 & a & 3 \\ 5 & b & -1 \\ 1 & c & 1 \end{bmatrix} \neq 0$$

The value of x is

A. $-b + 17c - 6a$

B. $5/(-b + 17c - 6a)$

C. 0

D. 1

E. $2b - 5a$

2. Let A be 6×3 matrix such that its null space is spanned by

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

The rank of A is

A. 1

B. 2

C. 3

D. 4

E. 5

3. Let A be a 3×1 matrix. Let B be a 2×3 matrix. Let C be a 2×1 matrix. Which of the following matrices is well defined and is a 1×2 matrix.

   A. AB

   B. $B^T C$

   C. $C^T BA$

   D. BAC^T

   E. $(BA)^T$

4. The dimension of the subspace of M_{22} consisting of vectors

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

satisfying

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$
$$3x_1 + 6x_2 + 9x_3 + 12x_4 = 0$$

is

A. 0.

B. 1.

C. 2.

D. 3.

E. 4.

5. Let A be an $n \times n$ nonsingular matrix. Which condition fails?

   E. 0 is an eigenvalue of A.

   B. $\det(A) \neq 0$.

   C. The rank of A is n.

   D. A^T is nonsingular.

   A. A^2 is nonsingular.

6. Let V be an inner product space and let $\mathbf{u}$ and $\mathbf{v}$ be two unit vectors which are perpendicular to each other. Then the value of $(3\mathbf{u} - 4\mathbf{v}, \mathbf{u} + 5\mathbf{v})$ is

   A. 17

   B. -17

   C. 3

   D. 20

   E. -60

7. Let a be a real number different from $-1, 0$ or 1, and let $A = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$. Which of the following vectors is an eigenvector for A?

A. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ a \end{bmatrix}$

E. $\begin{bmatrix} a \\ 1 \end{bmatrix}$

8. Which vector is in the orthogonal complement of the span of

$$\begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}-2\\3\\-1\end{bmatrix}, \begin{bmatrix}2\\2\\6\end{bmatrix}$$

A. $\begin{bmatrix}5\\-3\\8\end{bmatrix}$.

B. $\begin{bmatrix}-10\\2\\7\end{bmatrix}$.

C. $\begin{bmatrix}2\\3\\2\end{bmatrix}$.

D. $\begin{bmatrix}1\\2\\4\end{bmatrix}$.

E. $\begin{bmatrix}4\\-1\\6\end{bmatrix}$.

9. Which of the following transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$ is **not** linear?

A. $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

B. $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \end{bmatrix}$

C. $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ x+y \end{bmatrix}$

D. $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y \\ x+y \end{bmatrix}$

E. $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

10. Which of the given subsets of $\mathbb{R}^3$ is a subspace?

A. The set of all vectors of the form $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$.

B. The set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a \geq 0$.

C. The set of vectors of the form $\begin{bmatrix} a \\ a^2 \\ a^3 \end{bmatrix}$.

D. The set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a > b + c$.

E. The set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $2a - b + c = a$.

11. Let $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. Then A^5 is equal to

A. $-A$

B. A^2

C. $-A^2$

D. A^3

E. $-A^3$

12. Suppose that A and B are two 3×3 matrices satisfying $\det(A) = 9$ and $B^2 = A$. Determine the value of $\det(2AB^T A^{-1} B)$.

   A. 18.

   B. 72.

   C. 2.

   D. 8.

   E. 162.