1. It is given that $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$, $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Find the rank of A.
   (b) Find the nullity of A.
   (c) Find a basis for the column space of A. We require that you choose the vectors for the basis from the column vectors of A.
   (d) Find another basis for the column space of A.
   (e) Find a basis for the row space of A. We require that you choose the vectors for the basis from the row vectors of A.
   (f) Find another basis for the row space of A.
   (g) Find a basis for the null space of A.
   (h) Find a basis for the orthogonal complement of the row space of A.
   (i) Write the third column of A as a linear combination of the other columns.

2. It is given that $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 1 & -4 \\ -1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 3 & 6 & 2 & -5 \end{bmatrix}$, $\mathbf{rref}(A^T) = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Find the rank of A.
   (b) Find the nullity of A.
   (c) Find a basis for the column space of A. We require that you choose the vectors for the basis from the column vectors of A.
   (d) Find another basis for the column space of A.
   (e) Find a basis for the row space of A. We require that you choose the vectors for the basis from the row vectors of A.
   (f) Find another basis for the row space of A.
   (g) Find a basis for the null space of A.
   (h) Find a basis for the orthogonal complement of the row space of A.
   (i) Do the row vectors of A span $\mathbb{R}^4$?

3. Let a be a real number (parameter). Consider the linear system corresponding to the following augmented matrix for the system (variables are x, y, and z)

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2a & a & 1 \\ 1 & 2a & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

(a)Find all values of a for which the system has an unique solution.
(b) For what value of a does the system have the solution $x = 3$, $y = -1$, $z = 1$?

4. Determine the values of a so that the linear system

$$\begin{cases} x + \ \ y + \qquad\qquad z = 2 \\ 2x + 3y + \qquad\qquad z = 5 \\ 2x + 3y + (a^2 - 3)z = a + 3 \end{cases}$$

has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

5. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \ L = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \ L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}.$$

Find the standard matrix for L.

6. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \qquad L = \left(\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \qquad L = \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Write $\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(b) Find $L\left(\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}\right)$.

7. Find the standard matrix of the linear transformation L defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - z \\ x + 2y + z \\ 3x - y \end{bmatrix}.$$

8. If A is a 5×5 matrix and $\det A = 3$ find $A^2(\mathrm{adj}A)^2$ and $\det(A^2(\mathrm{adj}A)^2)$.

9. Compute the determinant.

(a) $\begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}$; (b) $\begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 2 \\ 4 & -2 & 1 \end{bmatrix}$.

10. (a) Find a basis for $S = \{t^2 + 2t, 3t^2 + t - 1\}$. Does $6t^2 - 1$ belong to $spanS$?

(b) Let $W = \left\{ \begin{bmatrix} a & b \\ 0 & 3b \end{bmatrix}; a, b \in \mathbb{R} \right\}$. Verify W is a subspace and find a basis for W. What is the dimension of W?

11. For what values of d are the vectors $\begin{bmatrix} 1 \\ 3 \\ d \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ linearly independent?

12. Given $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{bmatrix}$. Find adj A.

13. Find the inverse of the matrix $A = \begin{bmatrix} 2 & -2 & 1 \\ -3 & 2 & 0 \\ 4 & 1 & 1 \end{bmatrix}$.

14. Find all values of a for which the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 5 \\ 1 & a & 6 \end{bmatrix}$ is nonsingular.

15. Consider the set of all pairs (x, y) of real numbers, with the following operations:
   a) $(x, y) \oplus (w, z) = (x + w,\ y + z)$, for all pairs (x, y), (w, z).
   b) $c \odot (x, y) = (cx,\ y)$, for all pairs (x, y) and every number c.
   Is this a vector space? If not, state which conditions fail and show why they fail.

16. Suppose A is a 3×3 matrix with eigenvalues 1, 3, 5. What are the eigenvalues of A^3?
   Suppose $v = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$ is an eigenvector of A. Find an eigenvector of A^3.

17. Use the Gram-Schmidt process to construct an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \end{bmatrix} \right\}.$$

18. Let W be the subspace of $\mathbb{R}^3$ spanned by the vector $w = [1\ 1\ -1]$.
   (a) Find a basis for the orthogonal complement $W^\perp$ of W.

(b) Find an orthonormal basis for $W^\perp$.

19. Let W be the subspace of $\mathbb{R}^4$ spanned by the set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \end{bmatrix}$. (a) Find the orthogonal projection of $\mathbf{v}$ onto W. (b) Write $\mathbf{v}$ as $\mathbf{w} + \mathbf{u}$, with $\mathbf{w} \in W$ and $\mathbf{u} \in W^\perp$. (c) Find the distance from $\mathbf{v}$ to W.

20. Let A be a 5×7 matrix with $\operatorname{rank}(A) = 3$, and let $\mathbf{b}$ be a 5×1 matrix. Determine if each of the following statements is true.
    (a) For every $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is uniquely solvable. True False
    (b) For some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is uniquely solvable, and for some $\mathbf{b}$, it is not solvable. True False
    (c) For every $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. True False
    (d) For some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, and for some $\mathbf{b}$, it is not solvable. True False
    (e) For some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is not solvable. True False

21. A, B and C are 3×3 matrices and k is a scalar. determine if each of the following statements is *always* true.
    (a) If $A^2 = I_3$, then $A = I_3$ or $-I_3$. True False
    (b) $\det(AB) = \det(A)\det(B)$. True False
    (c) $\det(kA) = k^3 \det(A)$. True False
    (d) If C is invertible, the characteristic polynomial of A is the same as that of $C^{-1}AC$. True False
    (e) $(A - A^T)^T = A - A^T$. True False
    (f) $\det(\operatorname{adj}(A)) = \det(A)$. True False

22. For each of the following sets of vectors, determine if it is a vector (sub)space:
    (a) The set of all vectors in $\mathbb{R}^4$ with the property $2x_1 + x_2 - 3x_3 + x_4 = 0$; Yes No
    (b) The set of all matrices which have the form $\begin{bmatrix} a & b \\ c & 2 \end{bmatrix}$; Yes No
    (c) The set of all vectors in $\mathbb{R}^3$, which have the form $(0, a - b + c, 3b + c)$ where a, b and c are arbitrary real numbers; Yes No
    (d) The set of all polynomials P in the space of all polynomials of degree at most 7, with the property $P(2) = 0$; Yes No

(e) The set of all nonsingular matrices in the space of 3×3 matrices; Yes No

23. Determine if each of the following sets of vectors is linearly independent or linearly dependent:

(a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$; Independent Dependent

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$; Independent Dependent

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \right\}$; Independent Dependent

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} \right\}$; Independent Dependent

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Independent Dependent

24. We have a 3×3 matrix $A = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ with $\det(A) = 3$. Compute the determinant of the following matrix

(a) $\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} a & 1 & 1 \\ b & 3 & 3 \\ c & 5 & 5 \end{bmatrix}$

(d) A^T

(e) $(5A)^{-1}$

25. Find the determinants of the following matrices:

(a) $\begin{bmatrix} 1 & 3 & 2 & -1 \\ 10 & 2 & -1 & 3 \\ 7 & 0 & 10 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix}$.

(b) $\begin{bmatrix} 12 & 3 & 2 & 3 \\ 5 & 2 & -1 & 2 \\ 5 & 4 & 10 & 4 \\ 0 & 1 & 2 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 6 & 3 & -4 & 3 \\ 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 2 & 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 2 & 3 & 2 & -1 & 1 \\ 0 & 2 & -1 & 3 & 8 \\ 0 & 0 & 5 & 1 & -4 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

(e) $\begin{bmatrix} 2 & 3 & 2 & -1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

26. Find all eigenvalues of $A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

27. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.
    (a) Find all eigenvalues of A.
    (b) For each eigenvalue as above, find an eigenvector of A associated to it.
    (c) Find a diagonal matrix D and an orthogonal matrix P such that $P^TAP = D$.

28. Determine an invertible matrix $\hat{A}$ and a vector $\hat{b}$ such that the solution to $\hat{A}\hat{X} = \hat{b}$ is the least squares solution to $AX = b$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

29. Find the least squares fit line for the points

$$(1,2), (2,1), (3,3), (4,3).$$

30. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & -1 \\ 8 & 8 \end{bmatrix}$.

(a) Find all eigenvalues of A.
(b) Find the general solution to the linear system of differential equations.
(c) Solve the initial value problem for the given conditions $x_1(0) = 3$ and $x_2(0) = -10$.

31. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$.
(a) Find all eigenvalues of A.
(b) Find the real-valued general solution to the linear system of differential equations.
(c) Solve the initial value problem for the given conditions $x_1(0) = 5$ and $x_2(0) = 1$.

32. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$.
(a) Find all eigenvalues of A.
(b) Find the general solution to the linear system of differential equations.
(c) Solve the initial value problem for the given conditions

$$x_1(0) = -2,\ x_2(0) = 1,\ x_3(0) = 2$$

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