MATH 265 Exam 1, 09/23/2004

1. Consider the follow vectors in $\mathbf{R}^4$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 6 \\ 2 \\ 1 \\ -3 \end{bmatrix},$$

Determine whether $\mathbf{w}$ is in span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. If your answer is affirmative, give also the coefficients for expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. [*15pt*]

Solution. Assume $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{w}$. We get a system of linear equations with augmented matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ -1 & 1 & 1 & | & 2 \\ 1 & 1 & -1 & | & 1 \\ -1 & 1 & -1 & | & -3 \end{bmatrix}.$$

Gaussian elimination:

$$A \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 2 & 2 & | & 8 \\ 0 & 0 & -2 & | & -5 \\ 0 & 2 & 0 & | & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 1 & | & 4 \\ 0 & 0 & -2 & | & -5 \\ 0 & 2 & 0 & | & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 1 & | & 4 \\ 0 & 0 & -2 & | & -5 \\ 0 & 0 & -2 & | & -5 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 1 & | & 4 \\ 0 & 0 & 1 & | & 5/2 \\ 0 & 0 & 1 & | & 5/2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 1 & | & 4 \\ 0 & 0 & 1 & | & 5/2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

We get $a_3 = 5/2$, $a_2 = 3/2$, $a_1 = 2$. So $\mathbf{w}$ is indeed in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$: $\mathbf{w} = 5/2\mathbf{v}_1 + 3/2\mathbf{v}_2 + 2\mathbf{v}_3$.

2. Find the general solution to the system of linear equations with the following matrix as its agumented matrix: [*10pt*]

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 3 & | & 7 \\ 0 & 0 & 1 & 1 & 5 & | & 5 \\ 0 & 0 & 0 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & 2 & 2 & | & 6 \end{bmatrix}. \qquad \text{Answer: } \begin{cases} x_1 & = -2r + s + 5 \\ x_2 & = r \\ x_3 & = -4s + 2 \\ x_4 & = -s + 3 \\ x_5 & = s. \end{cases}$$

Solution. Gauss-Jordan reduction:

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 3 & | & 7 \\ 0 & 0 & 1 & 1 & 5 & | & 5 \\ 0 & 0 & 0 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 3 & | & 7 \\ 0 & 0 & 1 & 0 & 4 & | & 2 \\ 0 & 0 & 0 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & | & 5 \\ 0 & 0 & 1 & 0 & 4 & | & 2 \\ 0 & 0 & 0 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The variables x_1, x_3, x_4 are dependent, x_2, x_5 are free. Back substution gives $x_4 = -x_5+3$, $x_3 = -4x_5+2$, $x_1 = -2x_2 + x_5 + 5$. So the final answer is as show above.

3. Which ones of the following matrices are elementary matrices? Circle the elementary ones and write down their inverses in the space provided below. You don't have to do anything to the non-elementary ones. [*12pt*]

$$\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix},\quad \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 5 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix},\quad \begin{bmatrix}0 & 0 & 1\\0 & 2 & 0\\1 & 0 & 0\end{bmatrix},\quad \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\quad \begin{bmatrix}0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 0\end{bmatrix}.$$

Solution. An elementary matrix is obtained by doing a single row operation to the identity matrix of the same size, and we can get back the identity matrix by doing a single row operation to an elementary matrix. Therefore, the 1st, the 2nd, and the 5th are elementary. Their inverses are

$$\begin{bmatrix}1 & 0\\0 & 1/3\end{bmatrix},\quad \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & -5 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix},\quad \begin{bmatrix}0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}.$$

4. For each of the following parts, determine whether W is a subspace of V. Answer "is" or "is not". Give full justification to **two** of your answers in the space provided below.

(a) $V = \mathbf{R}^2$; $W =$ the subset of two vectors $\begin{bmatrix}x\\y\end{bmatrix}$ satisfying $xy = 0$. [*2pt*] **Answer:** is not

(b) $V = C^1(-\infty, \infty)$, continuously differentiable functions on the real line; $W =$ the subset of functions $f(x)$ satisfying $f'(-1) = 0$. [*2pt*] **Answer:** is

(c) $V = \mathbf{R}_4$; $W =$ the subset of matrices of the form $[a-b, b-c, c-d, d-a]$ for any real numbers a, b, c, d. [*2pt*] **Answer:** is

(d) $V = \mathbf{R}_3$; $W =$ the subset of matrices of the form $[b^3, b^2, b]$ for any real number b. [*2pt*] **Answer:** is not

(e) $V = M_{2\times 2}$; $W =$ the subset of matrices A satisfying $A^2 = \mathbf{0}$. [*2pt*] **Answer:** is not

Justification for part (a) [*4pt*]

Solution. Let $\mathbf{u} = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}0\\1\end{bmatrix}$, then $\mathbf{u}, \mathbf{v} \in W$ but $\mathbf{u} + \mathbf{v} = \begin{bmatrix}1\\1\end{bmatrix} \notin W$, not closed under addition.

Justification for part (b) [*4pt*]

Solution. If $f(x), g(x) \in W$, and c, d are real numbers, and $h(x) = cf(x) + dg(x)$, then $h'(-1) = cf'(-1) + dg'(-1) = 0$, so $h(x) \in W$ also.

Justification for part (c) [*4pt*]

Solution. The given expression is

$$a[1, 0, 0, -1] + b[-1, 1, 0, 0] + c[0, -1, 1, 0] + d[0, 0, -1, 1].$$

So W is the span of $[1, 0, 0, -1]$, $[-1, 1, 0, 0]$, $[0, -1, 1, 0]$, and $[0, 0, -1, 1]$.

Justification for part (d) [*4pt*]

Solution. Let $\mathbf{u} = [1, 1, 1] \in W$. Then $2\mathbf{u} = [2, 2, 2]$ is not of the form $[b^3, b^2, b]$. So $2u \notin W$. Thus W is not closed under scalar multiplication.

Justification for part (e) [*4pt*]

Solution. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $A^2 = B^2 = \mathbf{0}$ so $A, B \in W$ but $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin W$. So W is not closed under addition.

5. The following system of linear equations contains a parameter b. Determine all values of b so that the system has (a) a unique solution; (b) no solution; (c) infinitely many solutions. [*15pt*]

$$\begin{aligned} 2x + y + (b+2)z &= 5 \\ x + z &= 2 \\ x + y + (b^2+1)z &= b + 2 \end{aligned}$$

Solution. Gaussian reduction:

$$\begin{bmatrix} 2 & 1 & b+2 & 5 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & b^2+1 & b+2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & b+2 & 5 \\ 1 & 1 & b^2+1 & b+2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & b & 1 \\ 0 & 1 & b^2 & b \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & b & 1 \\ 0 & 0 & b^2-b & b-1 \end{bmatrix}$$

At this moment we need to know whether $b^2 - b = 0$ before proceeding. So we branch into cases: if $b^2 - b \neq 0$, i.e. $b \neq 0, 1$, the system clearly has a unique solution.

If $b = 0$, the last row becomes $[0, 0, 0, -1]$ which translates to the equation $0 = -1$, so there is no solution.

If $b = 1$, the last row becomes $[0, 0, 0, 0]$ and we see that x_3 will be a free variable. Therefore, there are infinitely many solutions.

Final answer: (a) when $b \neq 0, 1$, (b) when $b = 0$, (c) when $b = 1$.

6. Find the **reduced** row echelon form of the following matrix [*15pt*]

$$\begin{bmatrix} 0 & 1 & 2 & 4 & 2 & 4 \\ 1 & 2 & 5 & 11 & 2 & 15 \\ 1 & 1 & 3 & 7 & 0 & 11 \\ 2 & 4 & 10 & 22 & 5 & 29 \end{bmatrix}$$

Solution.

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 5 & 11 & 2 & 15 \\ 0 & 1 & 2 & 4 & 2 & 4 \\ 1 & 1 & 3 & 7 & 0 & 11 \\ 2 & 4 & 10 & 22 & 5 & 29 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 5 & 11 & 2 & 15 \\ 0 & 1 & 2 & 4 & 2 & 4 \\ 0 & -1 & -2 & -4 & -2 & -4 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 5 & 11 & 2 & 15 \\ 0 & 1 & 2 & 4 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 5 & 11 & 2 & 15 \\ 0 & 1 & 2 & 4 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 5 & 11 & 0 & 17 \\ 0 & 1 & 2 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 3 & 0 & 5 \\ 0 & 1 & 2 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{Answer}$$

7. (a) Find the inverse of the following matrix [*10pt*]

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 7 & 2 \\ 0 & 3 & 6 & 1 \end{bmatrix}. \qquad \textit{Answer:} A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 5 & 1 & -2 \\ 0 & -4 & -1 & 2 \\ 0 & 9 & 3 & -5 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 7 & 2 & 0 & 0 & 1 & 0 \\ 0 & 3 & 6 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & -3 & 0 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & -2/5 & 1/5 & 0 \\ 0 & 0 & 3 & 1 & 0 & -3 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & -2/5 & 1/5 & 0 \\ 0 & 0 & 0 & -1/5 & 0 & -9/5 & -3/5 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & -2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 9 & 3 & -5 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -4 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 9 & 3 & -5 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 4 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 5 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & -4 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 9 & 3 & -5 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 5 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & -4 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 9 & 3 & -5 \end{array}\right].$$

The final answer is shown above.

(b) Find the inverse of A^T. [*5pt*]

Solution. This is just $(A^{-1})^T$, so

Answer:

$$(A^T)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 5 & -4 & 9 \\ 0 & 1 & -1 & 3 \\ 0 & -2 & 2 & -5 \end{bmatrix}.$$