ME 323 Examination #2 **SOLUTION**

April 11, 2018

PROBLEM NO. 1 – 25 points max.

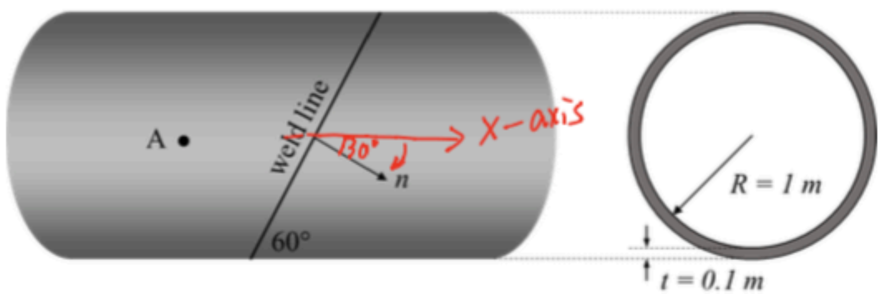
A thin-walled pressure vessel is fabricated by welding together two, open-ended stainless-steel vessels along a 60° weld line. The welded vessel has an internal radius of $R = 1\ m$ and a thickness $t = 0.1\ m$. The gas pressure inside the vessel is p.

a) Determine the axial and hoop stresses, σ_a and σ_h, in terms of the internal pressure p. Draw the state of stress at point A and its Mohr's circle.

b) Use the Mohr's circle to determine the components of stress, σ_n, σ_t and τ_{nt}, in terms of the internal pressure p, along the weld line.

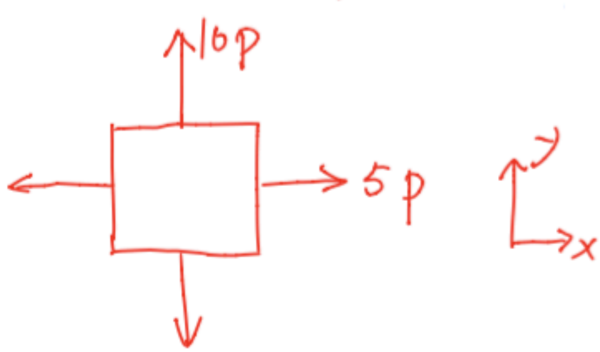
c) Use the Mohr's circle to determine the maximum gas pressure p, such that BOTH of the following conditions are met:

- The maximum shear stress of the vessel does not exceed the yield strength of stainless-steel, $\sigma_Y = 1000\ MPa$, AND:
- The maximum normal stress along the weld line does not exceed the welding strength in tension, $\sigma_{wt} = 3000\ MPa$.

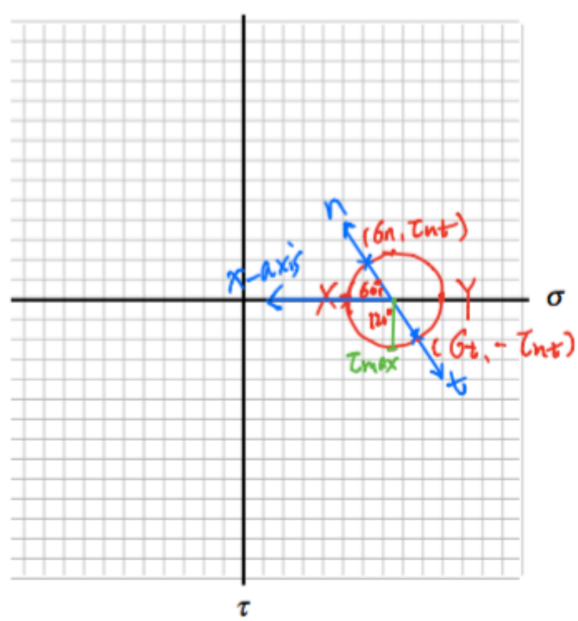


$$\sigma_a = \frac{PR}{2t} = \frac{p \cdot 1m}{2 \times 0.1m} = 5p$$

$$\sigma_h = \frac{PR}{t} = \frac{p \cdot 1m}{0.1m} = 10p$$



$X(5p, 0)$ $Y(10p, 0)$



$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{5p + 10p}{2} = 7.5p$$

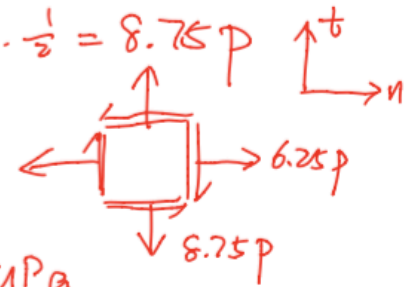
$$R = \sqrt{\left(\frac{5p - 10p}{2}\right)^2 + 0} = 2.5p$$

(2) After CW rotate 30°

$$\sigma_n = \sigma_{ave} - R \cdot \cos 60^\circ = 7.5p - 2.5p \cdot \frac{1}{2} = 6.25p$$

$$\tau_{nt} = -R \sin 60^\circ = -2.165p$$

$$\sigma_t = \sigma_{ave} + R \cdot \cos 60^\circ = 7.5p + 2.5p \cdot \frac{1}{2} = 8.75p$$



(3).

I. Max. in-plan shear stress

$$\tau_{max} = R = 2.5p < 1000 \text{ MPa}$$

$$p < 400 \text{ MPa}$$

II. Max. normal stress along the weld line

$$\sigma_t = 8.75p < 3000 \text{ MPa}$$

$$p < 343 \text{ MPa}$$

Max p shoud be 343 MPa

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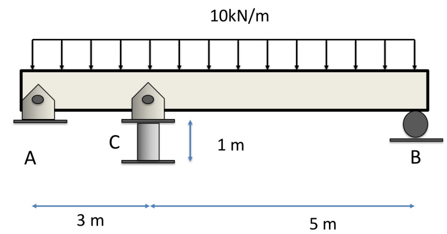
PROBLEM NO. 2 – 25 points max.

The beam AB is supported by a pin at A, a roller at B and a post having a 20 cm diameter at C. The post and the beam have a Young's modulus of $E = 200$ GPa, and the second area moment of the beam is $I = 200\ 10^6$ mm^4. In your analysis, neglect the contribution of shear strain to the total strain energy.



a) Using Castigliano's second theorem:

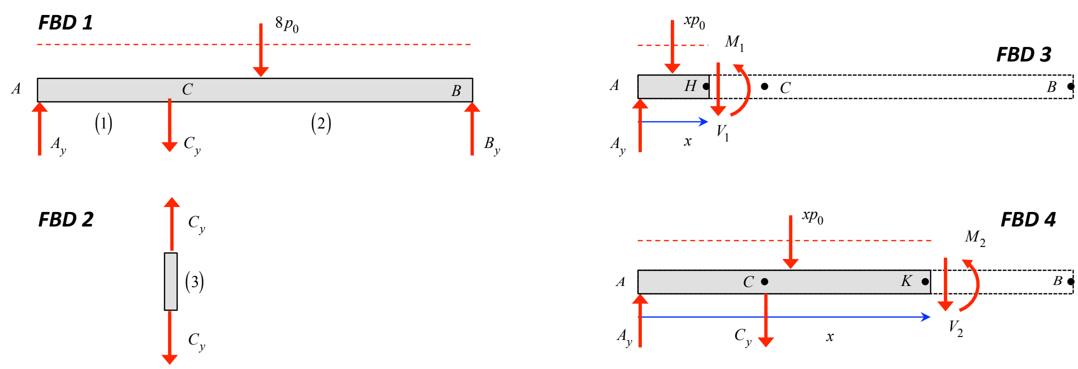
   i. Determine the reactions on the beam at A, B and C.

   ii. Determine the slope of the beam deflection at B.

b) Under what assumptions can you neglect the contribution of shear strain to the total strain energy?

   If the dimensions of the beam cross section are much smaller than the length of the beam.

SOLUTION



<u>*From FBD 1*</u>

$$\sum M_A = -C_y(3) + B_y(8) - 8p_0(4) = 0 \ \Rightarrow\ B_y = \frac{3}{8}C_y + 4p_0$$

$$\sum F_y = A_y + B_y - C_y - 8p_0 = 0 \ \Rightarrow\ A_y = \frac{5}{8}C_y + 4p_0$$

The problem is *INDETERMINATE*: will choose C_y as the redundant load.

<u>*From FBDs 3 and 4:*</u>

$$\sum M_h = -A_y x + x p_0\left(\frac{x}{2}\right) + M_1 = 0 \ \Rightarrow\ M_1(x) = \left(4p_0 + \frac{5}{8}C_y\right)x - \frac{1}{2}p_0 x^2$$

$$\sum M_K = -A_y x + x p_0\left(\frac{x}{2}\right) + C_y(x-3) + M_2 = 0 \ \Rightarrow\ M_2(x) = \left(-\frac{3}{8}C_y + 4p_0\right)x + 3C_y - \frac{1}{2}p_0 x^2$$

METHOD #1 – consider the strain energy from only the beam (C_y is an external "applied" force)

$$U = \frac{1}{2EI}\int_0^3 M_1^2(x)\,dx + \frac{1}{2EI}\int_3^8 M_2^2(x)\,dx$$

Using Castigliano's theorem, along with rod elongation equation: $v_C = C_y / EA$:

$$-v_C = \frac{\partial U}{\partial C_y} \;\Rightarrow\; -\frac{C_y}{EA} = \frac{1}{EI}\int_0^3 M_1 \frac{\partial M_1}{\partial C_y} dx + \frac{1}{EI}\int_3^8 M_2 \frac{\partial M_1}{\partial C_y} dx \;\Rightarrow$$

$$0 = \frac{1}{EI}\int_0^3 \left(\frac{5}{8}x\right)\left[\left(4p_0 + \frac{5}{8}C_y\right)x - \frac{1}{2}p_0 x^2\right]dx + \frac{1}{EI}\int_3^8 \left(-\frac{3}{8}x + 3\right)\left[\left(-\frac{3}{8}C_y + 4p_0\right)x + 3C_y - \frac{1}{2}p_0 x^2\right]dx + \frac{C_y}{EA}$$

$$= \frac{1}{EI}\int_0^3 \left[\left(\frac{5}{2}p_0 + \frac{25}{64}C_y\right)x^2 - \frac{5}{16}p_0 x^3\right]dx$$

$$+ \frac{1}{EI}\int_3^8 \left[9C_y + \left(-\frac{9}{4}C_y + 12p_0\right)x + \left(\frac{9}{64}C_y - 3p_0\right)x^2 + \frac{3}{16}p_0 x^3\right]dx + \frac{C_y}{EA}$$

$$= \frac{1}{EI}\left[\frac{1}{3}\left(\frac{5}{2}p_0 + \frac{25}{64}C_y\right)x^3 - \frac{5}{64}p_0 x^4\right]_{x=0}^{x=3}$$

$$+ \frac{1}{EI}\left[9C_y x + \frac{1}{2}\left(-\frac{9}{4}C_y + 12p_0\right)x^2 + \frac{1}{3}\left(\frac{9}{64}C_y - 3p_0\right)x^3 + \frac{3}{64}p_0 x^4\right]_{x=3}^{x=8} + \frac{C_y}{EA}$$

$$= \frac{1}{EI}\left[\frac{1}{3}\left(\frac{5}{2}p_0 + \frac{25}{64}C_y\right)(27) - \frac{5}{64}p_0(81)\right]$$

$$+ \frac{1}{EI}\left[9C_y(8-3) + \frac{1}{2}\left(-\frac{9}{4}C_y + 12p_0\right)(64-9) + \frac{1}{3}\left(\frac{9}{64}C_y - 3p_0\right)(512-27) + \frac{3}{64}p_0(4096-81) + \frac{I}{A}C_y\right]$$

where: $\frac{I}{A} = \frac{\left(200\times10^6 mm^4\right)\left(m/1000mm\right)^4}{\pi\left(0.2/2\right)^2 m^2} = \frac{0.02}{\pi}m^2$

Solving gives

$C_y = -49.3\ kN$

$B_y = \frac{3}{8}C_y + 4p_0 = 21.5\ kN$

$A_y = \frac{5}{8}C_y + 4p_0 = 9.18\ kN$

METHOD #2 – consider the strain energy from both the beam and rod together (C_y is a reaction force)

$$U = \frac{1}{2EI}\int_0^3 M_1^2(x)dx + \frac{1}{2EI}\int_3^8 M_2^2(x)dx + \frac{1}{2}\frac{C_y^2(1)}{EA}$$

Using Castigliano's theorem:

$$0 = \frac{\partial U}{\partial C_y} = \frac{1}{EI}\int_0^3 M_1\frac{\partial M_1}{\partial C_y}dx + \frac{1}{EI}\int_3^8 M_2 \frac{\partial M_1}{\partial C_y}dx + \frac{C_y}{EA}$$

This gives the <u>same</u> equation for C_y as does METHOD #1.

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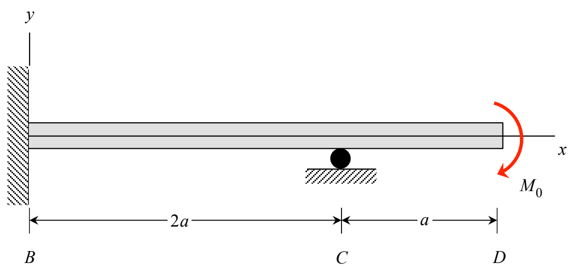
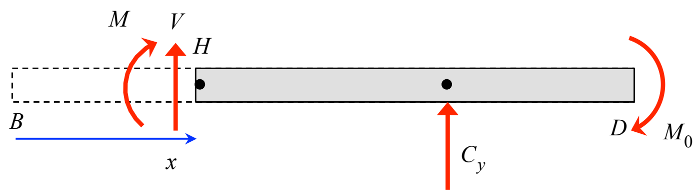
PROBLEM NO. 3 – 25 points max.

Beam BD is supported by a fixed wall at end B, and by a roller support at C. A concentrated couple M_0 acts at D. The beam is made up of a material with a Young's modulus E and has a constant cross section with a second area moment of I over the full length of the beam. Using either the *second-order* or *fourth-order integration integration methods*:

a) determine the *reaction force* acting on the beam at C.
b) determine the slope of the deflection θ_D of the beam at D

Leave your answers in terms of, at most: M_0, E, I and a.





Section BC

$$\sum M_H = -M(x) + C_y(2a - x) - M_0 = 0 \quad \Rightarrow \quad M(x) = (2aC_y - M_0) - C_y x$$

$$\theta(x) = \theta(0) + \frac{1}{EI}\int_0^x \left[(2aC_y - M_0) - C_y x\right]dx = 0 + \frac{1}{EI}\left[(2aC_y - M_0)x - \frac{1}{2}C_y x^2\right]$$

$$v(x) = v(0) + \frac{1}{EI}\int_0^x \left[(2aC_y - M_0)x - \frac{1}{2}C_y x^2\right]dx = 0 + \frac{1}{EI}\left[\frac{1}{2}(2aC_y - M_0)x^2 - \frac{1}{6}C_y x^3\right]$$

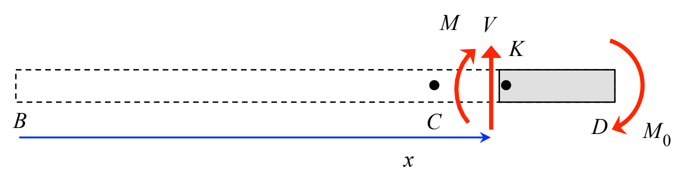
Since $v(2a) = 0$, we have:

$$0 = \frac{1}{EI}\left[\frac{1}{2}(2aC_y - M_0)(2a)^2 - \frac{1}{6}C_y(2a)^3\right] = \frac{1}{EI}\left[\frac{8}{3}C_y a^3 - 2M_0 a^2\right] \quad \Rightarrow \quad \boxed{C_y = \frac{3}{4}\frac{M_0}{a}}$$

Also,

$$\theta_C = \theta(2a) = \frac{1}{EI}\left[\left(2a\left(\frac{3}{4}\frac{M_0}{a}\right) - M_0\right)(2a) - \frac{1}{2}\left(\frac{3}{4}\frac{M_0}{a}\right)(2a)^2\right] = \frac{1}{EI}\left[aM_0 - \frac{3}{2}aM_0\right] = -\frac{1}{2}\frac{aM_0}{EI}$$

Section CD



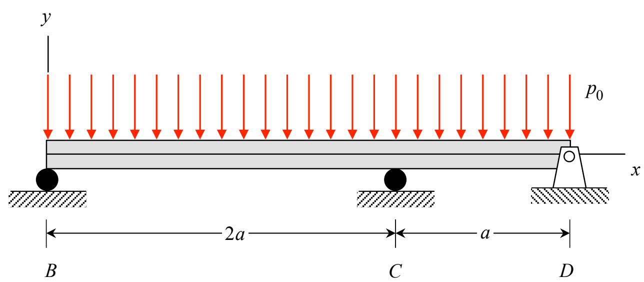
$$\sum M_K = -M_0 - M(x) = 0 \quad \Rightarrow \quad M(x) = -M_0$$

Therefore,

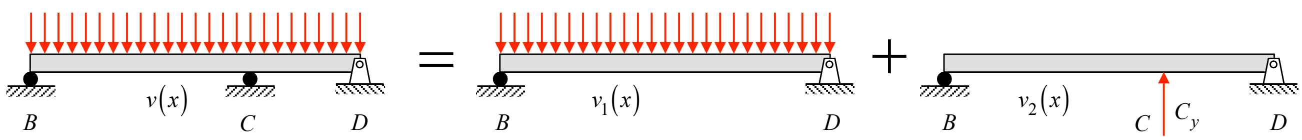
$$\theta(3a) = \theta(2a) + \frac{1}{EI}\int_{2a}^{3a}\left[-M_0\right]dx = -\frac{1}{2}\frac{M_0}{EI}a - \frac{M_0}{EI}(3a - 2a) = \boxed{-\frac{3}{2}\frac{M_0}{EI}a = \theta_D}$$

PROBLEM NO. 4 - PART A – 7 points max.

A beam is made up a material with a Young's modulus of E and has a constant cross section with a second area moment of I. A downward, constant line load p_0 (force/length) acts along the full length of the beam. The beam has roller supports at B and C, along with a pin joint support at end D. Using the *superposition* approach, determine the reaction force acting on the beam at the roller support C.



SOLUTION



$$v(2a) = 0 = v_1(2a) + v_2(2a)$$

$$= -\frac{1}{24}(2a)\left[(3a)^3 - 2(3a)(2a)^2 + (2a)^3\right]\frac{p_0}{EI} + \frac{1}{6}(a)(2a)\left[(3a)^2 - a^2 - (2a)^2\right]\frac{C_y}{EI(3a)}$$

$$= \frac{a^3}{EI}\left[-\frac{11}{12}p_0 a + \frac{4}{9}C_y\right] \Rightarrow C_y = \frac{33}{16}p_0 a$$

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PROBLEM NO. 4 - PART B – 3 points max.

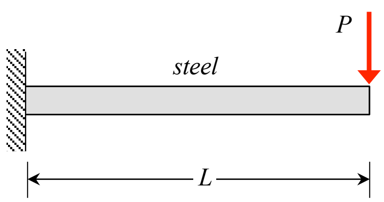
Cantilevered beams A, B and C shown below are acted upon by a point load P acting at the free end of the beam. The cross sections and lengths of each beam are the same. Beams A and B are made up of steel and aluminum, respectively, whereas Beam C is made up of both steel and aluminum components over its length. Let $|\sigma_A|_{max}$, $|\sigma_B|_{max}$ and $|\sigma_C|_{max}$ be the maximum normal stress in Beams A, B and C, respectively. Circle the correction responses below:

TRUE or FALSE: $|\sigma_A|_{max} = |\sigma_B|_{max}$

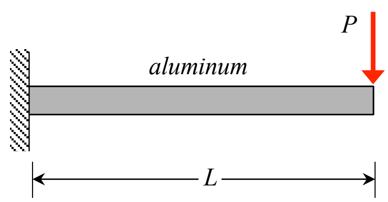
TRUE or FALSE: $|\sigma_A|_{max} = |\sigma_C|_{max}$

TRUE or FALSE: $|\sigma_B|_{max} = |\sigma_C|_{max}$

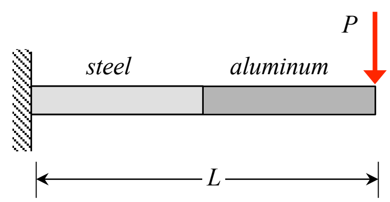
All structures are DETERMINATE.
Stresses do not depend on materials.



Beam A



Beam B



Beam C

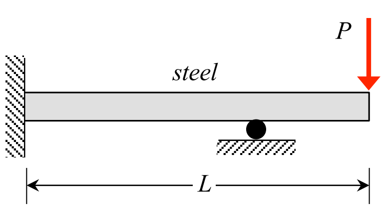
PROBLEM NO. 4 - PART C – 2 points max.

The proppped-cantilevered beams A, B and C shown below are acted upon by a point load P acting at the free end of the beam. The cross sections and lengths of each beam are the same. Beams A and B are made up of steel and aluminum, respectively, whereas Beam C is made up of both steel and aluminum components over its length. Let $|\sigma_A|_{max}$, $|\sigma_B|_{max}$ and $|\sigma_C|_{max}$ be the maximum normal stress in Beams A, B and C, respectively. Circle the correction responses below:

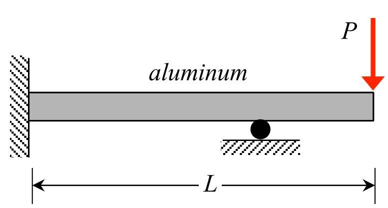
TRUE or FALSE: $|\sigma_A|_{max} = |\sigma_C|_{max}$

TRUE or FALSE: $|\sigma_B|_{max} = |\sigma_C|_{max}$

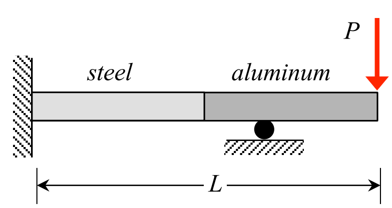
All structures are INDETERMINATE.
Stresses depend on deflections, which are dependent on material properties.



Beam A



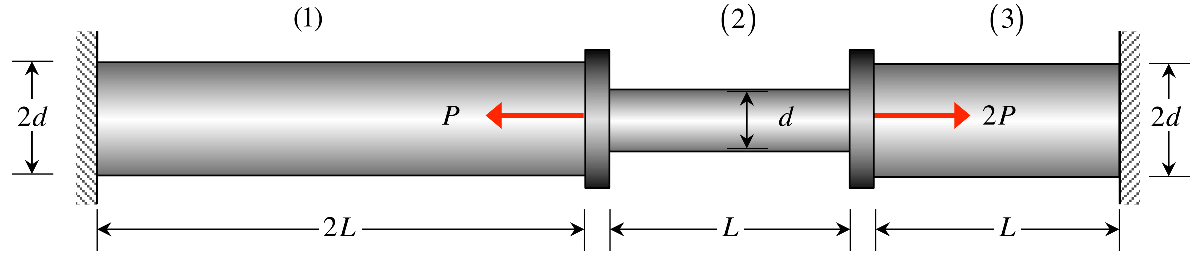
Beam B



Beam C

PROBLEM NO. 4 - PART D – 7 points max.

A rod is made up of solid, circular cross-section segments (1), (2) and (3), where Segment (1) has a length of $2L$ and diameter $2d$, Segment (2) has a length of L and diameter d, and Segment (3) has a length of L and diameter $2d$. All segments are made up of a material having a Young's modulus of E. Loads of P and 2P act on the rigid connectors, as shown below. You are asked to set up a three element, finite element model for displacement analysis of this rod, using one element for each of the rod segments. To this end, write down the stiffness matrix $\left[K\right]$ and load vector $\left\{F\right\}$ for the model *after* the boundary conditions have been enforced on the model.

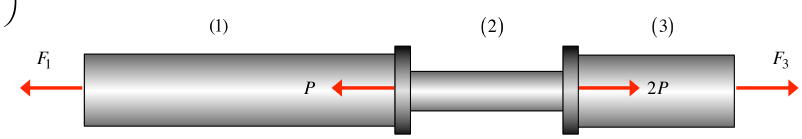


SOLUTION

$$k_1 = \frac{EA_1}{L_1} = \frac{E\pi(2d/2)^2}{2L} = \frac{\pi}{2}\frac{Ed^2}{L} = 2\left(\pi\frac{Ed^2}{4L}\right)$$

$$k_2 = \frac{EA_2}{L_2} = \frac{E\pi(d/2)^2}{L} = \frac{\pi}{4}\frac{Ed^2}{L} = \pi\frac{Ed^2}{4L}$$

$$k_3 = \frac{EA_3}{L_3} = \frac{E\pi(2d/2)^2}{L} = \pi\frac{Ed^2}{L} 4\left(\pi\frac{Ed^2}{4L}\right)$$



Therefore, before enforcing BCs:

$$\left[K\right] = \pi\frac{Ed^2}{4L}\begin{bmatrix} 2 & -2 & & \\ -2 & 3 & -1 & \\ & -1 & 5 & -4 \\ & & -4 & 4 \end{bmatrix} ; \qquad \left\{F\right\} = \begin{Bmatrix} -F_1 \\ -P \\ 2P \\ F_3 \end{Bmatrix}$$

After enforcing BCs (removing 1st and 4th rows and columns of [K] and the 1st and 4th row of {F}:

$$\left[K\right] = \pi\frac{Ed^2}{4L}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} ; \qquad \left\{F\right\} = \begin{Bmatrix} -P \\ 2P \end{Bmatrix}$$

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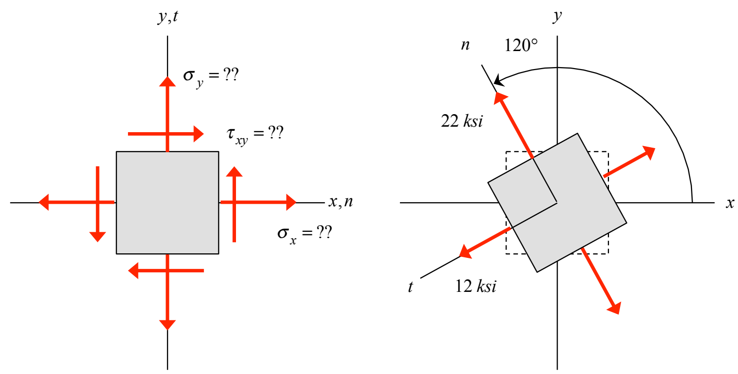
PROBLEM NO. 4 - <u>PART E</u> – 6 points max.

A state of stress is characterized by its unknown *x-y* components on the stress element shown below. When the stress element is rotated through an angle of $120°$, the state of stress has the *n-t* components shown below right.

a) Draw the Mohr's circle for this state of stress on the axes provided below. Carefully label the center of the circle as well as its radius. Show the x-axis on the Mohr's circle.

b) What is the maximum in-plane shear stress?

c) What are the σ_x, σ_y and τ_{xy} components of this state of stress?



SOLUTION

$$\sigma_{ave} = \frac{\sigma_{P1}+\sigma_{P2}}{2} = \frac{22+12}{2} = 17\ ksi$$

$$R = \frac{\sigma_{P1}-\sigma_{P2}}{2} = \frac{22-12}{2} = 5\ ksi$$

$$|\tau|_{max,in-plane} = R = 5\ ksi$$

From Mohr's circle shown:

$$\sigma_x = \sigma_{ave} - R\cos 60° = 17-(5)\cos 60° = 14.5\ ksi$$

$$\sigma_y = \sigma_{ave} + R\cos 60° = 17+(5)\cos 60° = 19.5\ ksi$$

$$\tau_{xy} = -R\sin 60° = -(5)\sin 60° = -4.33\ ksi$$