November 14, 2017

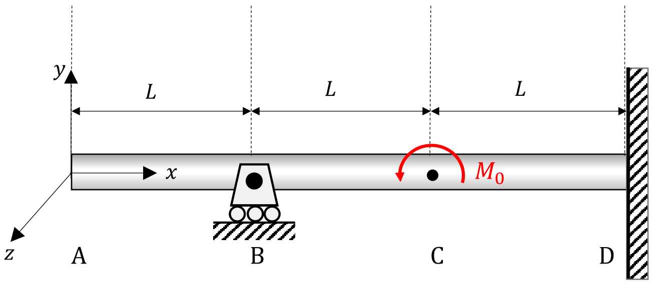
PROBLEM NO. 1 – 30 points max.

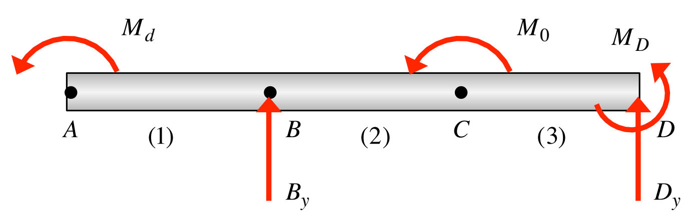
The cantilever beam AD of the bending stiffness EI is subjected to a concentrated moment M_0 at C. The beam is also supported by a roller at B. Using Castigliano's theorem:

a) Determine the reaction force at the roller B.

b) Determine the rotation angle of the beam about z axis at the end A.

Ignore the shear energy due to bending. Express your answers in terms of M_0, E, and I.





SOLUTION

External reactions

Using FBD of entire beam:

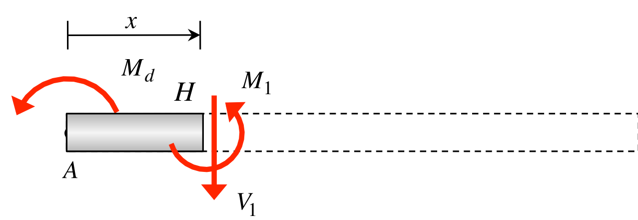
$$\sum M_D = M_0 - B_y(2L) + M_d + M_D = 0$$

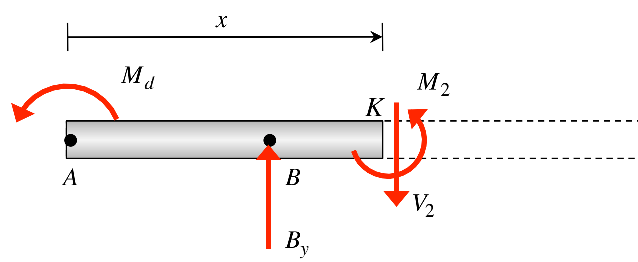
$$\sum F_y = B_y + D_y = 0$$

Problem is INDETERMINATE. Will choose B_y as the redundant reaction:

$$M_D = 2B_yL - M_0 - M_d$$

$$D_y = -B_y$$





Strain energy

From FBD with cut through section (1):

$$\sum M_H = M_1 + M_d = 0 \;\Rightarrow\; M_1 = -M_d$$

From FBD with cut through section (2):

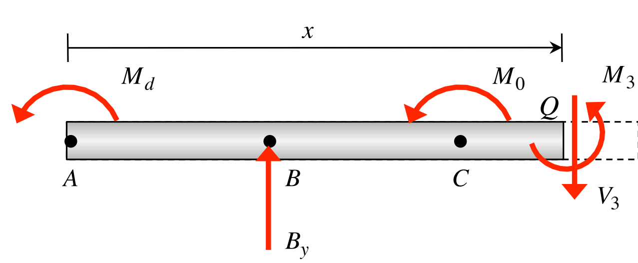
$$\sum M_H = M_2 + M_d - B_y(x-L) = 0 \;\Rightarrow$$

$$M_2 = -M_d + B_y(x-L)$$

From FBD with cut through section (3):

$$\sum M_H = M_3 + M_d - B_y(x-L) + M_0 = 0 \;\Rightarrow$$

$$M_3 = -M_d + B_y(x-L) - M_0$$



From this, we have:

$$U = U_1 + U_2 + U_3$$

$$= \frac{1}{2EI}\int_0^L \left[-M_d\right]^2 dx + \frac{1}{2EI}\int_L^{2L}\left[-M_d + B_y\left(x-L\right)\right]^2 dx + \frac{1}{2EI}\int_{2L}^{3L}\left[-M_d + B_y\left(x-L\right) - M_0\right]^2 dx$$

<u>Castigliano's theorem</u>

Since B_y is our redundant reaction, we can write:

$$0 = \left[\frac{\partial U}{\partial B_y}\right]_{M_d=0}$$

$$= 0 + \frac{1}{EI}\int_L^{2L}\left[-M_d + B_y\left(x-L\right)\right]_{M_d=0}\left(x-L\right)dx + \frac{1}{EI}\int_{2L}^{3L}\left[-M_d + B_y\left(x-L\right) - M_0\right]_{M_d=0}\left(x-L\right)dx$$

$$= \frac{B_y}{EI}\left[\int_L^{2L}\left(x^2 - 2Lx + L^2\right)dx + \frac{1}{EI}\int_{2L}^{3L}\left(x^2 - 2Lx + L^2\right)dx\right] - \frac{M_0}{EI}\int_{2L}^{3L}\left(x-L\right)dx$$

$$= \frac{B_y}{EI}\left\{\frac{1}{3}\left[\left(2L\right)^3 - L^3\right] - L\left[\left(2L\right)^2 - L^2\right] + L^2\left(2L-L\right) + \frac{1}{3}\left[\left(3L\right)^3 - \left(2L\right)^3\right] - L\left[\left(3L\right)^2 - \left(2L\right)^2\right] + L^2\left(3L-2L\right)\right\}$$

$$- \frac{M_0}{EI}\left\{\frac{1}{2}\left[\left(3L\right)^2 - \left(2L\right)^2\right] - L\left[3L-2L\right]\right\}$$

$$= \frac{B_y L^3}{EI}\left\{\frac{7}{3} - 3 + 1 + \frac{19}{3} - 5 + 1\right\} - \frac{M_0 L^2}{EI}\left\{\frac{3}{2}\right\}$$

$$= \frac{8}{3}\frac{B_y L^3}{EI} - \frac{3}{2}\frac{M_0 L^2}{EI}$$

Therefore:

$$B_y = \frac{9}{16}\frac{M_0}{L}$$

Also:

$$\theta_A = \left[\frac{\partial U}{\partial M_d}\right]_{M_d=0}$$

$$= \frac{1}{EI}\left[\int_0^L M_d\,dx\right]_{M_d=0} + \frac{1}{EI}\int_L^{2L}\left[-M_d + B_y(x-L)\right]_{M_d=0}(-1)\,dx + \frac{1}{EI}\int_{2L}^{3L}\left[-M_d + B_y(x-L) - M_0\right]_{M_d=0}(-1)\,dx$$

$$= 0 - \frac{B_y}{EI}\int_L^{2L}(x-L)\,dx - \frac{1}{EI}\int_{2L}^{3L}\left[B_y(x-L) - M_0\right]dx$$

$$= -\frac{B_y}{EI}\left\{\frac{1}{2}\left[(2L)^2 - L^2\right] - L(2L-L)\right\} - \frac{B_y}{EI}\left\{\frac{1}{2}\left[(3L)^2 - (2L)^2\right] - L(3L-2L)\right\} + \frac{M_0}{EI}(3L-2L)$$

$$= -\frac{B_y L^2}{EI}\left\{\frac{3}{2}-1\right\} - \frac{B_y L^2}{EI}\left\{\frac{5}{2}-1\right\} + \frac{M_0 L}{EI}$$

$$= -2\frac{B_y L^2}{EI} + \frac{M_0 L}{EI}$$

$$= -2\left(\frac{9}{16}\frac{M_0}{L}\right)\frac{L^2}{EI} + \frac{M_0 L}{EI}$$

$$= -\frac{1}{8}\frac{M_0 L}{EI}$$

ME 323 Examination #2 **SOLUTION**

November 14, 2017

PROBLEM NO. 2 – 25 points max.

At a point A above the neutral axis of the beam shown in the figure, the state of plane stress can be described by the insert on the right-hand side of the figure. The maximum in-plane shear stress at this point is $\tau_{max} = 13$MPa, the normal stress in the x-direction is $\sigma_x = 20$MPa, and normal stress in the y-direction is $\sigma_y = 0$MPa.

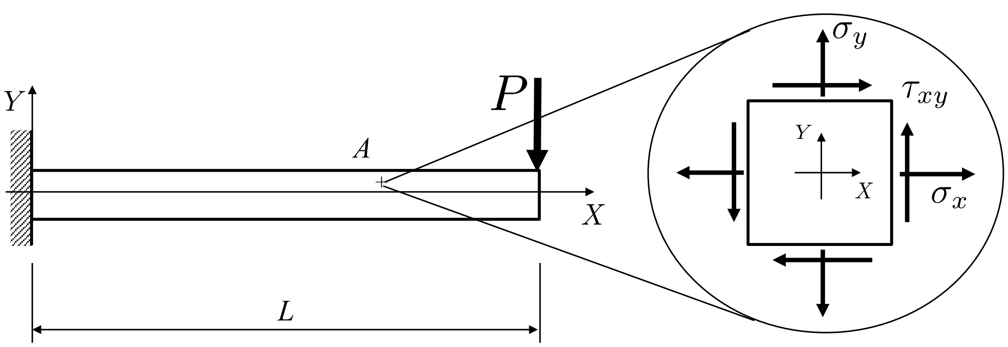
a) Determine the magnitude of the shear stress, τ_{xy}, on the x and y faces.

b) Determine the sign of the shear stress τ_{xy}. HINT: Determine first the direction of the shear force acting on the cross-section with normal x at point A.

c) Draw Mohr's circle corresponding to the state of stress at point A. Clearly indicate the location of the center of the circle, the radius of the circle and point X (which represents the stress state on the x-face) in this drawing.

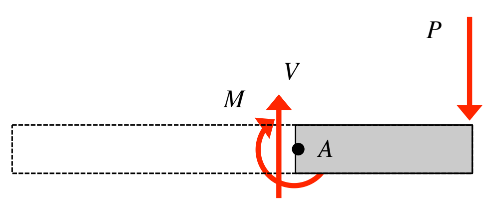
d) Determine the two in-plane principal stresses at this point. Determine the rotation angle of the stress element for each principal stress.

e) Show the locations of the principal stresses and of the in-plane maximum shear stress on your Mohr's circle in c) above.



State of stress at point A

SOLUTION



Internal resultants

$$\sum F_y = -P + V = 0 \;\Rightarrow\; V = P$$

Stress transformation results:

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{20}{2} = 10\ MPa$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{20}{2}\right)^2 + \tau^2} = \sqrt{100 + \tau^2}$$

Since:

$$|\tau|_{max,in-plane} = R = 13$$

we can write:

$$\sqrt{100+\tau^2} = 13 \;\Rightarrow\; \tau = \pm\sqrt{13^2 - 100} = \pm\sqrt{69}$$

Choose the "-" sign based on the direction of the applied force P.

Mohr's circle and principal stresses

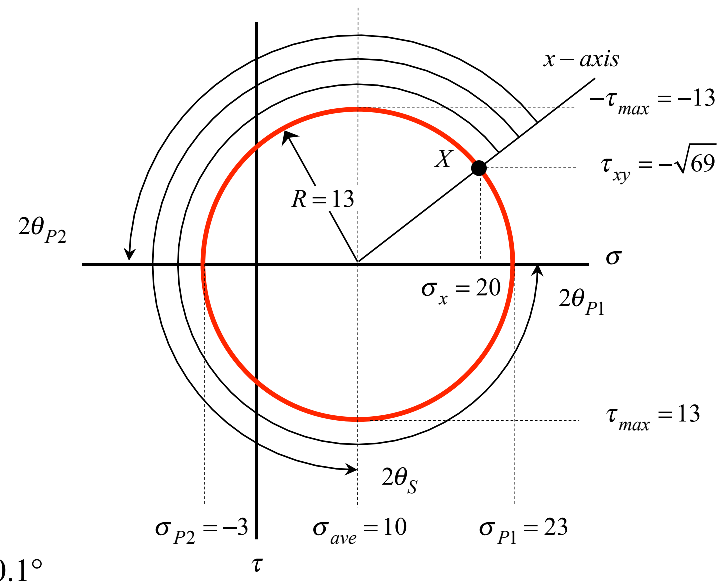
The Mohr's circle is centered at $(\sigma_{ave}, 0) = (10, 0)\ MPa$ and has a radius of $R = 13\ MPa$, as shown. From this, the principal components of stress are:

$$\sigma_{P1} = \sigma_{ave} + R = 10 + 13 = 23\ MPa$$

$$\sigma_{P2} = \sigma_{ave} - R = 10 - 13 = -3\ MPa$$

The location X of the x-axis on Mohr's circle is:

$$\left(20, -\sqrt{69}\right)\ MPa\ .$$



From the figure, we see that the rotation angles from the x-axis to the above principal components of stress are:

$$2\theta_{P2} = 180 - tan^{-1}\left(\frac{\sqrt{69}}{20-10}\right) = \;\Rightarrow\; \theta_{P2} = 70.1°$$

$$2\theta_{P1} = 2\theta_{P2} + 180° \;\Rightarrow\; \theta_{P1} = \theta_{P2} + 90° = 160.1°$$

$$2\theta_S = 2\theta_{P2} + 90° \;\Rightarrow\; \theta_S = \theta_{P2} + 45° = 115.1°$$

PROBLEM NO. 3 – 25 points max.

The propped cantilever in the figure is simply supported at end A and fixed at end B. It supports a linearly distributed load of maximum intensity w_0 on the span AB.

a) Draw a free body diagram of the structure. Assume the reactions forces act in the direction of positive x and y axes, and the reaction moments act counterclockwise.

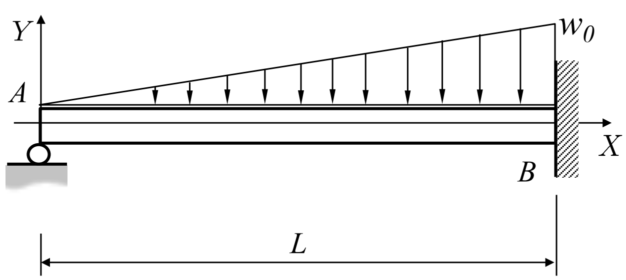
b) State the equations of equilibrium of the structure and indicate whether it is statically determinate or indeterminate.

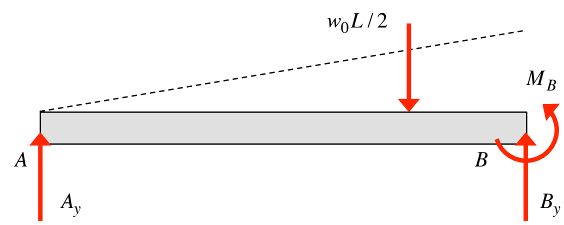
c) Indicate all the boundary conditions that correspond to this problem.

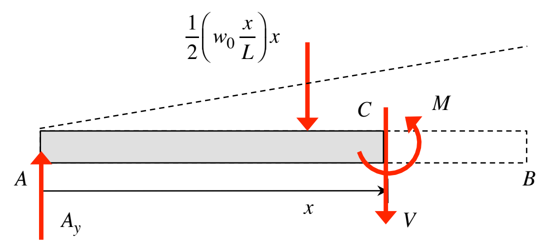
d) Use the second-order integration method (or the fourth-order integration method) to determine an expression for the reaction(s) at the support A. Express the result as a *sole* function of L, w_0 and EI.

e) Determine an expression for the deflection of the beam as a *sole* function of L, w_0 and EI.

f) Sketch the deflection curve.







SOLUTION

Equilibrium

From FBD of entire beam:

$$\sum M_A = -\left(\frac{1}{2}w_0 L\right)\left(\frac{2}{3}L\right) + B_y L + M_B = 0 \;\Rightarrow\; B_y L + M_B = \frac{1}{3}w_0 L^2$$

$$\sum F_y = A_y + B_y - \frac{1}{2}w_0 L = 0 \;\Rightarrow\; A_y + B_y = \frac{1}{2}w_0 L$$

From FBD with cut through beam at location "x":

$$\sum M_C = -A_y x + \frac{1}{2}\left(w_0 \frac{x^2}{L}\right)\left(\frac{1}{3}x\right) + M = 0 \;\Rightarrow M(x) = A_y x - \frac{1}{6}\frac{w_0 x^3}{L}$$

Integrations

Will need to enforce the following boundary conditions: $v(0) = v(L) = \theta(L) = 0$.

$$\theta(x) = \theta(0) + \frac{1}{EI}\int_0^x M(x)\,dx = \theta_A + \frac{1}{EI}\int_0^x \left(A_y x - \frac{1}{6}\frac{w_0 x^3}{L} \right) dx$$

$$= \theta_A + \frac{1}{EI}\left[\frac{1}{2}A_y x^2 - \frac{1}{24}\frac{w_0 x^4}{L} \right]$$

$$v(x) = v(0) + \int_0^x \theta(x)\,dx = 0 + \int_0^x \left[\theta_A + \frac{1}{EI}\left(\frac{1}{2}A_y x^2 - \frac{1}{24}\frac{w_0 x^4}{L} \right) \right] dx$$

$$= \theta_A x + \frac{1}{EI}\left[\frac{1}{6}A_y x^3 - \frac{1}{120}\frac{w_0 x^5}{L} \right]$$

Enforcing the boundary conditions at B:

$$0 = \theta(L) = \theta_A + \frac{1}{EI}\left[\frac{1}{2}A_y L^2 - \frac{1}{24}w_0 L^3 \right] \;\Rightarrow\; \theta_A = \frac{1}{EI}\left[-\frac{1}{2}A_y L^2 + \frac{1}{24}w_0 L^3 \right]$$

$$0 = v(L) = \theta_A L + \frac{1}{EI}\left[\frac{1}{6}A_y L^3 - \frac{1}{120}w_0 L^4 \right] \;\Rightarrow\; \theta_A = \frac{1}{EI}\left[-\frac{1}{6}A_y L^2 + \frac{1}{120}w_0 L^3 \right]$$

Equating the above two expressions for θ_A:

$$\frac{1}{EI}\left[-\frac{1}{2}A_y L^2 + \frac{1}{24}w_0 L^3 \right] = \frac{1}{EI}\left[-\frac{1}{6}A_y L^2 + \frac{1}{120}w_0 L^3 \right] \;\Rightarrow$$

$$-\frac{1}{2}A_y + \frac{1}{24}w_0 L = -\frac{1}{6}A_y + \frac{1}{120}w_0 L \;\Rightarrow$$

$$\left[\frac{1}{2} - \frac{1}{6} \right] A_y = \left[\frac{1}{24} - \frac{1}{120} \right] w_0 L \;\Rightarrow\; A_y = \frac{w_0 L}{10}$$

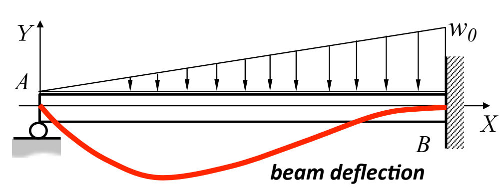
and:

$$\theta_A = \frac{1}{EI}\left[-\frac{1}{2}A_y L^2 + \frac{1}{24}w_0 L^3 \right] = \frac{1}{EI}\left[-\frac{1}{2}\left(\frac{w_0 L}{10} \right) L^2 + \frac{1}{24}w_0 L^3 \right] = -\frac{1}{120}\frac{w_0 L^3}{EI}$$

Therefore:

$$v(x) = -\frac{1}{120}\left(\frac{w_0 L^3}{EI} \right) x + \frac{1}{EI}\left[\frac{1}{6}\left(\frac{w_0 L}{10} \right) x^3 - \frac{1}{120}w_0 x^4 \right]$$

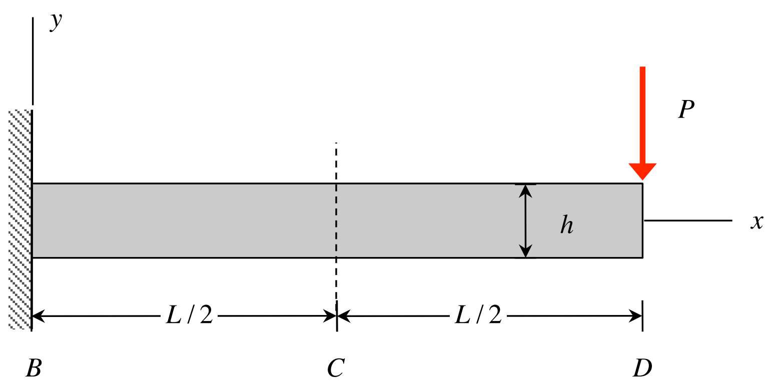
$$= \frac{w_0 L^4}{EI}\left[-\frac{1}{120}\left(\frac{x}{L} \right) + \frac{1}{60}\left(\frac{x}{L} \right)^3 - \frac{1}{120}\left(\frac{x}{L} \right)^4 \right]$$

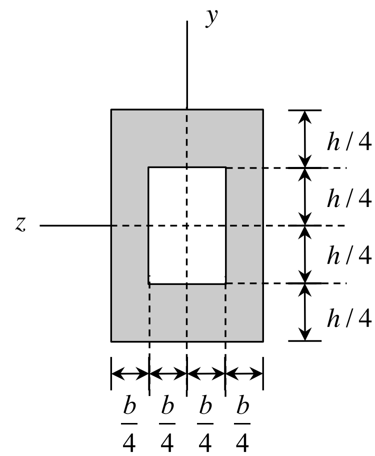


ME 323 Examination #2 **SOLUTION**

November 14, 2017

PROBLEM NO. 4 - PART A – 4 points max.





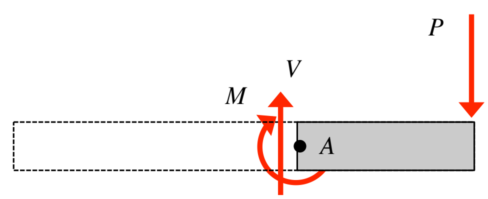
beam cross section at C

Consider the cantilevered beam above with the concentrated load P at end D. Determine the *shear stress* on the neutral surface of the beam at location C along the beam.

SOLUTION

Internal resultant shear force

$$\sum F_y = -P + V = 0 \quad \Rightarrow \quad V = P$$



Shear stress

$$\tau = \frac{VA^*\bar{y}^*}{It}$$

with:

$$A^*\bar{y}^* = \left(\frac{b}{4}\right)\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) + \left(\frac{b}{2}\right)\left(\frac{h}{4}\right)\left(\frac{3h}{8}\right) + \left(\frac{b}{4}\right)\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) = \frac{7}{64}bh^2$$

$$I = \frac{1}{12}bh^3 - \frac{1}{12}\left(\frac{b}{2}\right)\left(\frac{h}{2}\right)^3 = \frac{15}{192}bh^3$$

$$t = 2\left(\frac{b}{4}\right) = \frac{b}{2}$$

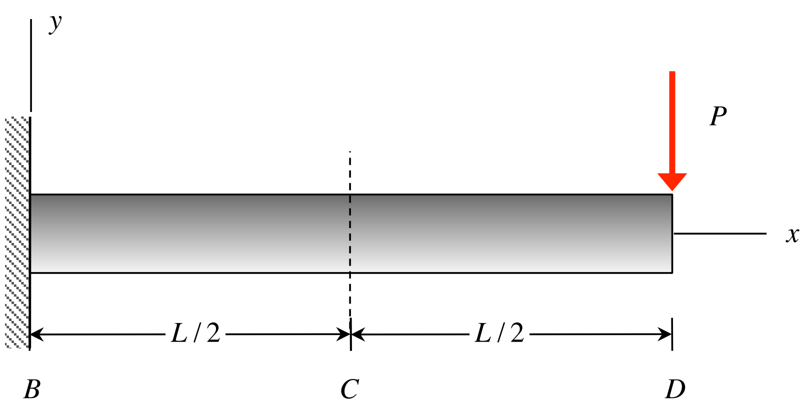
Therefore,

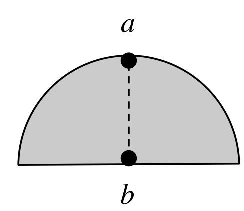
$$\tau = \frac{P\left[7bh^2/64\right]}{\left[15bh^3/192\right]\left[b/2\right]} = \frac{14}{5}\frac{P}{bh}$$

ME 323 Examination #2 **SOLUTION**

November 14, 2017

PROBLEM NO. 4 - PART B – 3 points max.





beam cross section at C

Consider the cantilevered beam above with the concentrated load P at end D. Consider the axial components of stress at points "a" and "b" (σ_a and σ_b, respectively) at location C along the beam. Circle the response below that most accurately describes the relative sizes of the magnitudes of these two stresses:

a) $|\sigma_a| > |\sigma_b|$

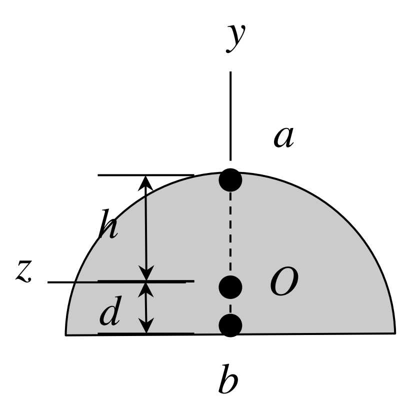
b) $|\sigma_a| = |\sigma_b|$

c) $|\sigma_a| < |\sigma_b|$

SOLUTION

Let O be the centroid of the cross section. Therefore,

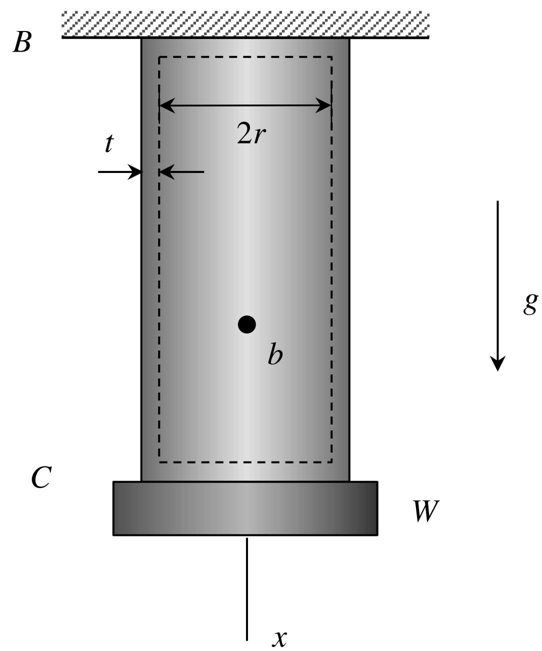
$$\frac{|\sigma_a|}{|\sigma_b|} = \frac{Mh / I}{Md / I} = \frac{h}{d} > 1 \quad \Rightarrow \quad |\sigma_a| > |\sigma_b|$$



ME 323 Examination #2 **SOLUTION**

November 14, 2017

PROBLEM NO. 4 - PART C – 4 points max.



Consider the thin-walled pressure vessel above that contains a gas under a pressure of p. The vessel is attached to a fixed support at B and has a plate of weight W attached to it at end C. Ignore the weight of the vessel. Determine the weight W of the plate for which the *maximum in-plane shear stress* in the vessel at point "b" is *zero*.

SOLUTION

$$\sigma_h = \frac{pr}{t}$$

$$\sigma_a = \frac{pr}{2t} + \frac{W}{2\pi rt}$$

For zero maximum in-plane shear stress:

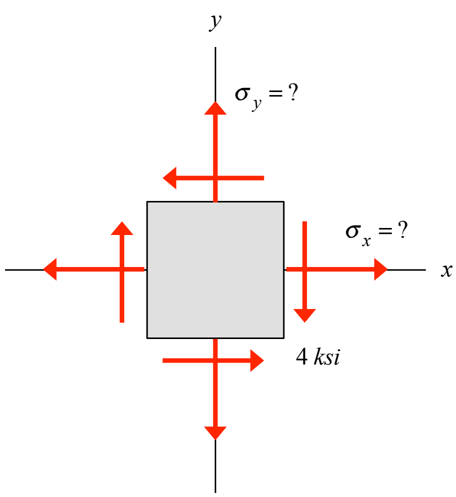
$$R = 0 = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{\sigma_h - \sigma_a}{2}\right)^2 + 0} = \frac{\sigma_h - \sigma_a}{2} \Rightarrow \sigma_h = \sigma_a \Rightarrow$$

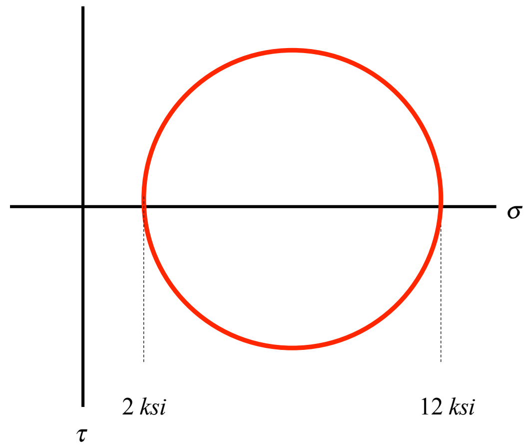
$$\frac{pr}{t} = \frac{pr}{2t} + \frac{W}{2\pi rt} \Rightarrow W = \pi pr^2$$

ME 323 Examination #2 **SOLUTION**

November 14, 2017

PROBLEM NO. 4 - PART D – 4 points max.





Consider the state of plane stress shown above left where the two normal components of stress, σ_x and σ_y, are unknown. The Mohr's circle for this state of stress is provided in the figure above right. Determine numerical values for the two normal components of stress σ_x and σ_y. There may be more than one set of answers; you need only find one set.

SOLUTION
From the above Mohr's circle, we see that:

$$\sigma_{ave} = \frac{12+2}{2} = 7\ ksi$$

$$R = \frac{12-2}{2} = 5\ ksi$$

Therefore:

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \quad \Rightarrow \quad \sigma_x + \sigma_y = 2\sigma_{ave} = 14\ ksi$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \Rightarrow \quad \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 = R^2 \quad \Rightarrow$$

$$\sigma_x - \sigma_y = \pm 2\sqrt{R^2 - \tau_{xy}^2} = \pm 2\sqrt{5^2 - (-4)^2} = \pm 6ksi$$

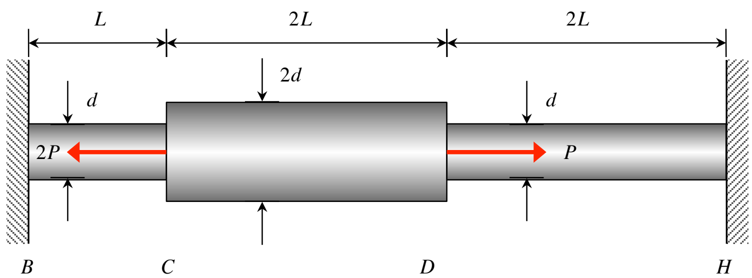
Choosing the "+" sign and solving gives: $\sigma_x = 10\ ksi$ and $\sigma_y = 4\ ksi$.

Choosing the "-" sign and solving gives: $\sigma_x = 4\ ksi$ and $\sigma_y = 10\ ksi$.

ME 323 Examination #2 **SOLUTION**

November 14, 2017

PROBLEM NO. 4 - PART E – 5 points max.



A rod is made up of three circular cross-section components: BC,CD and DH. The material for all components have a Young's modulus of *E*. Suppose you are to develop a finite element model for the rod using one element for each component. If the equilibrium equations after the enforcement of boundary conditions are to be written as:

$$[K]\{u\}=\{F\}$$

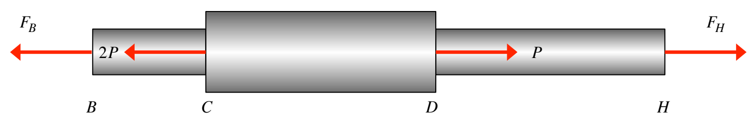
determine the stiffness matrix $[K]$ and the load vector $\{F\}$.

Spring stiffnesses:

$$k_1=\frac{E\pi(d/2)^2}{L}=\frac{\pi}{4}\frac{Ed^2}{L}$$

$$k_2=\frac{E\pi(2d/2)^2}{2L}=\frac{\pi}{2}\frac{Ed^2}{L}$$

$$k_3=\frac{E\pi(d/2)^2}{2L}=\frac{\pi}{8}\frac{Ed^2}{L}$$



Stiffness and forcing

Therefore the global stiffness matrix and forcing vector before enforcing BCs are:

$$[K]=\begin{bmatrix} 2 & -2 & & \\ -2 & 6 & -4 & \\ & -4 & 5 & -1 \\ & & -1 & 1 \end{bmatrix}\frac{\pi Ed^2}{8L} \quad \text{and} \quad \{F\}=\begin{Bmatrix} -F_B \\ -2P \\ P \\ F_H \end{Bmatrix}$$

Eliminating the first and last row and column of [K] and the first and last row of {F}:

$$[K]=\begin{bmatrix} 6 & -4 \\ -4 & 5 \end{bmatrix}\frac{\pi Ed^2}{8L} \quad \text{and} \quad \{F\}=\begin{Bmatrix} -2P \\ P \end{Bmatrix}$$