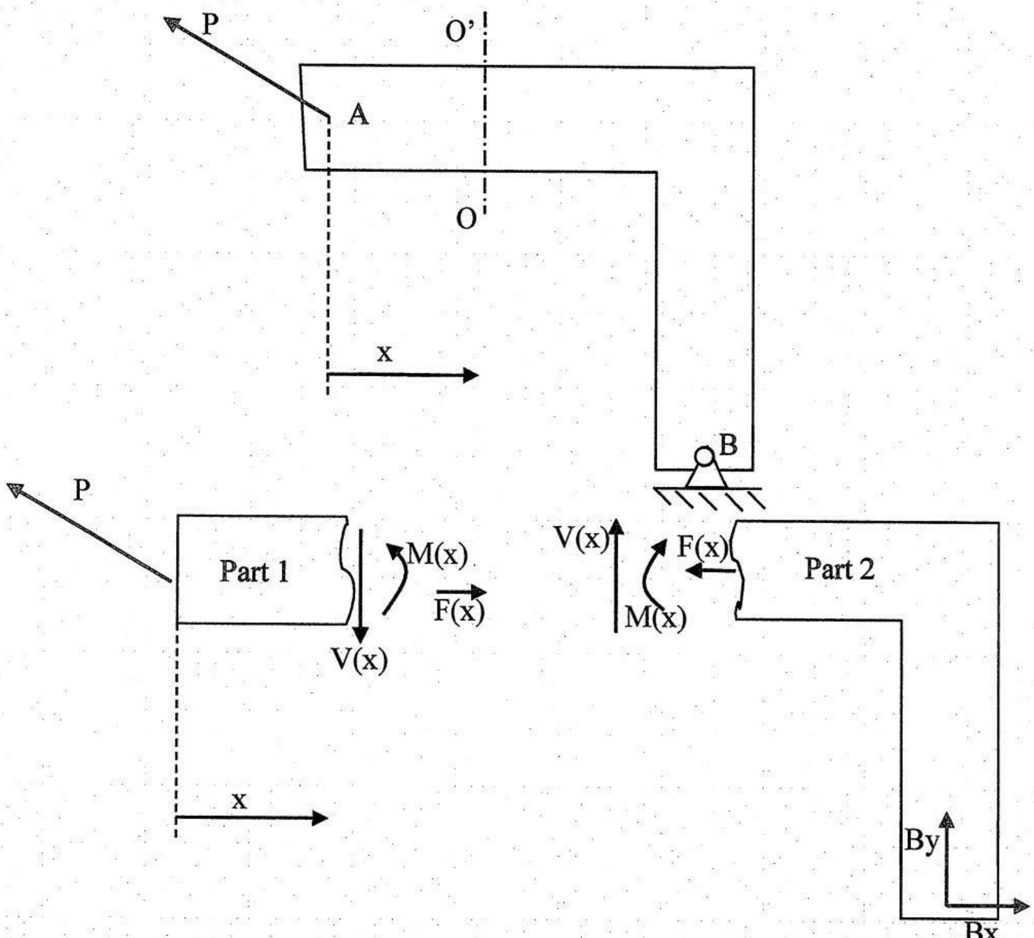
ME 323 Review

Internal resultants

What are they?

- Consider a structure at equilibrium under the action of external forces (red).
- Make a virtual section at any location within the body, say OO', separate the two parts of the body.
- Ask the question what internal forces/moments (green) does one part apply to the other to keep the whole structure together in equilibrium?



Internal resultants at a section in the body are <u>all</u> the resultant forces and moments that the remaining part of the body applies to given part in order to keep the whole body together. They can be classified as:

(a) Internal axial force resultant (F)
(b) Internal shear force resultant (V)
(c) Internal bending moment (M)
(d) Internal torque (T) (not shown above)

<u>Important notes</u>

1. Internal resultants depend on the location and orientation of the section (i.e depend on x)
2. Internal resultants applied to part 1by part 2 must be equal and opposite to internal resultants applied to part 2 by part 1.

3D stress and Strain and Hooke's law

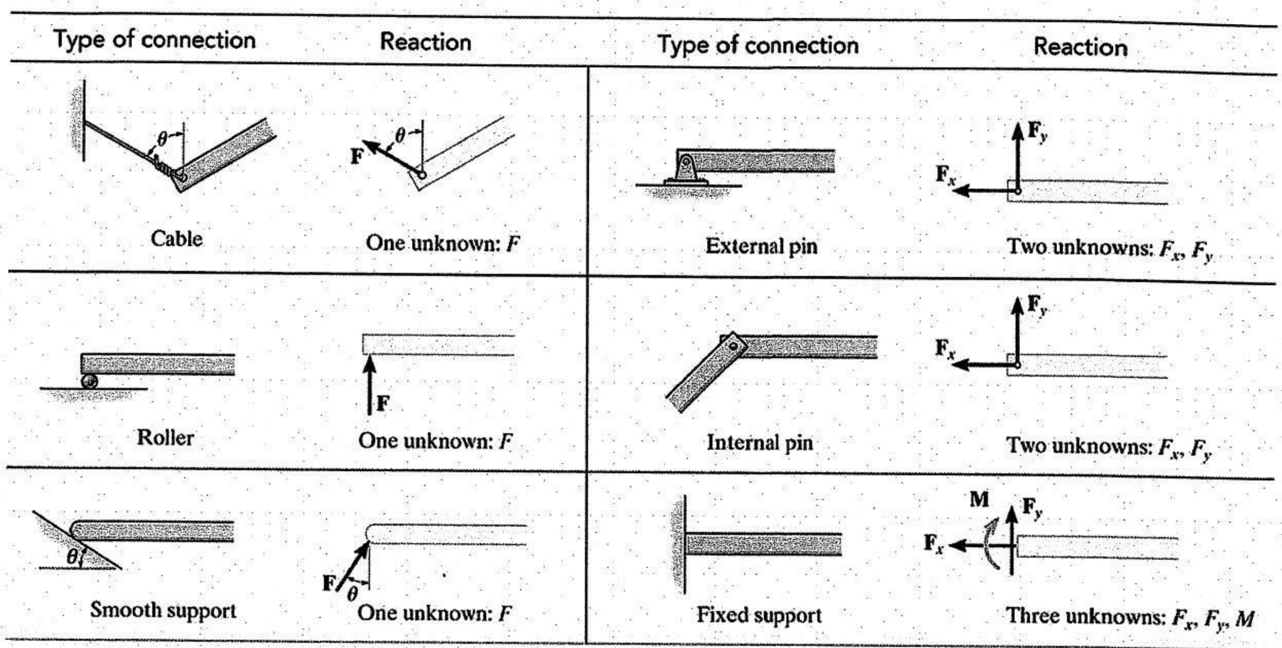
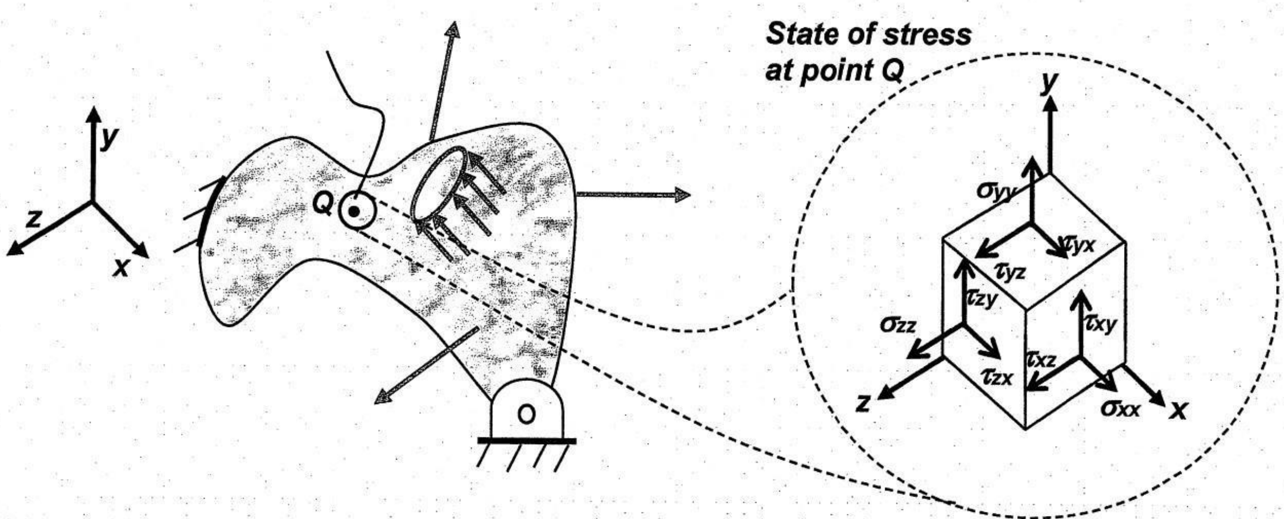
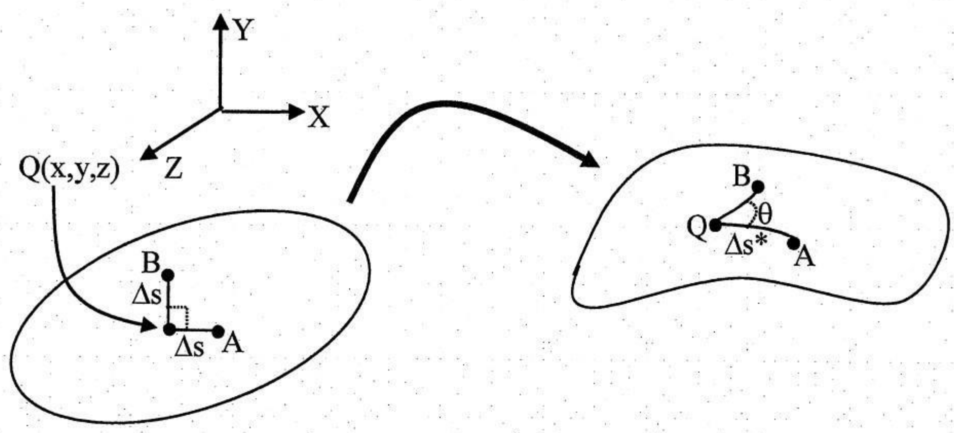


Figure: 01-01-02T

ME 323 Review

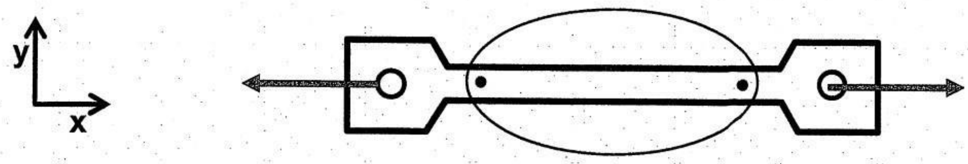


At each point in the interior and surface of a body under external loads, there exist 3 normal stress components ($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$) and 3 independent shear stress components ($\tau_{xy}, \tau_{yz}, \tau_{zx}$) referred to along the Cartesian XYZ coordinate system.



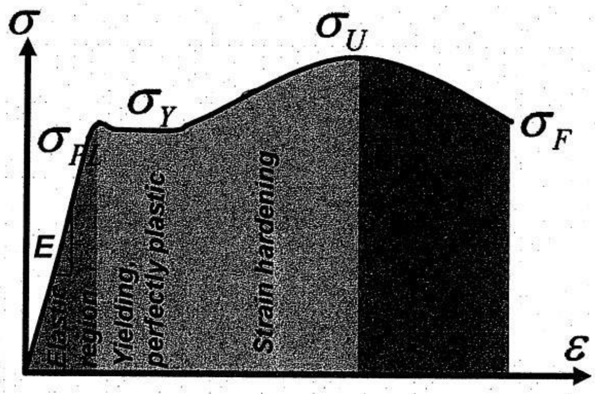
At each point in the interior and surface of a body under external loads, there exist 3 normal stress components ($\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$) and 3 independent shear stress components ($\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$) referred to along the Cartesian XYZ coordinate system.

ME 323 Review

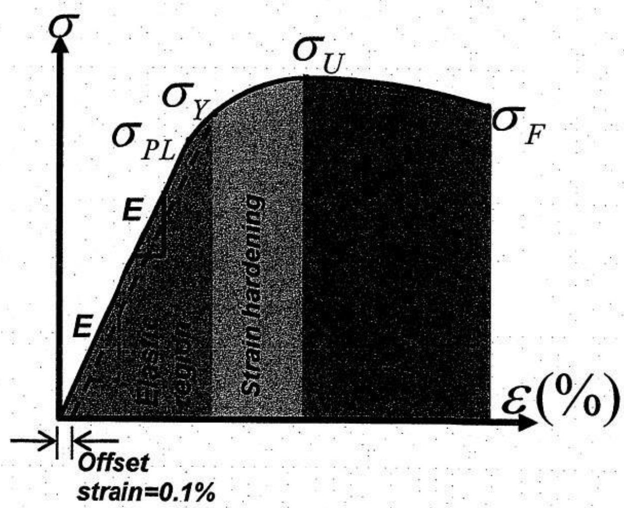


Material properties are tested using "dogbone" specimens in a tensile testing machine. In a tensile testing machine only one normal stress component σ_{xx} in the body is non-zero. This state of stress is called "one-dimensional" or "uniaxial".

Structural steel



Aluminum



- *Strength of a material* refers either to its Yield stress (σ_Y) or its Ultimate stress (σ_U) or its failure stress (σ_F).
- *Stiffness of a material* refers to its Young's modulus (E).
- *Ductility of a material* is the (plastic) strain at failure.
- *Toughness of a material* refers to the area under the stress-strain curve in the plastic region.

Hooke's law in three dimensions with thermal effects

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \upsilon\frac{\sigma_{yy}}{E} - \upsilon\frac{\sigma_{zz}}{E} + \alpha\Delta T = \frac{1}{E}\left[\sigma_{xx} - \upsilon(\sigma_{yy} + \sigma_{zz})\right] + \alpha\Delta T$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \upsilon\frac{\sigma_{xx}}{E} - \upsilon\frac{\sigma_{zz}}{E} + \alpha\Delta T = \frac{1}{E}\left[\sigma_{yy} - \upsilon(\sigma_{xx} + \sigma_{zz})\right] + \alpha\Delta T$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \upsilon\frac{\sigma_{xx}}{E} - \upsilon\frac{\sigma_{yy}}{E} + \alpha\Delta T = \frac{1}{E}\left[\sigma_{zz} - \upsilon(\sigma_{xx} + \sigma_{yy})\right] + \alpha\Delta T$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \quad \gamma_{yz} = \frac{1}{G}\tau_{yz} \quad \gamma_{zx} = \frac{1}{G}\tau_{zx}$$

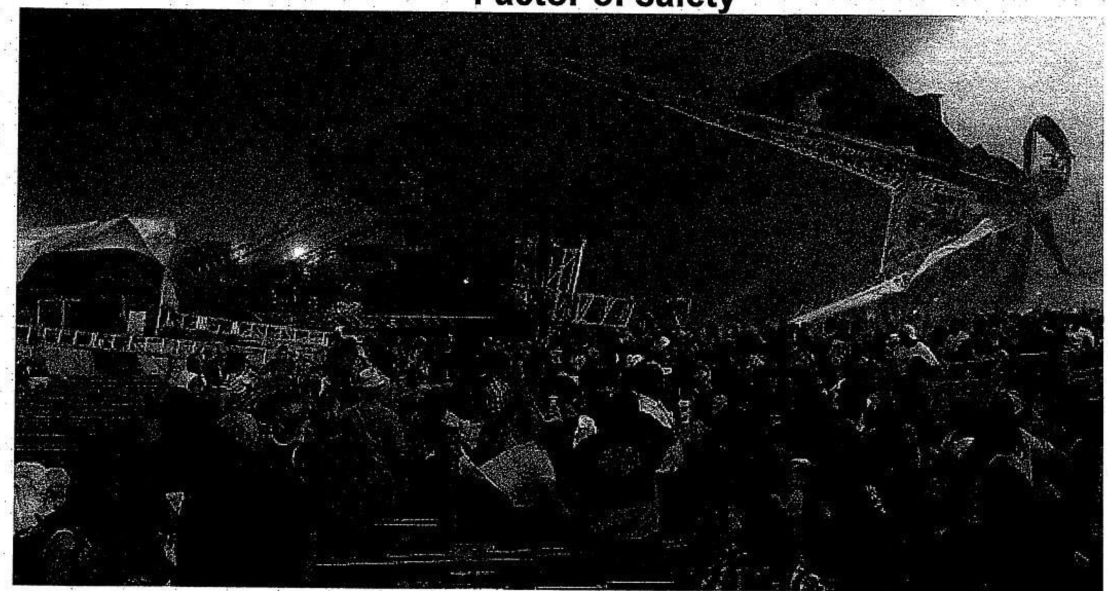
where E and ν are the Young's modulus and Poisson's ratio respectively, α is the coefficient of thermal expansion.

G is the shear modulus of the material which can be shown (Section 2.11 of Craig's book) to be related to E and ν as follows

$$G = \frac{E}{2(1+\upsilon)}$$

ME 323 Review

Factor of safety



Indiana State Fair stage collapse

An engineer must ensure that the designed structure can withstand the expected loads without yielding or failing. However both the material properties and expected loads are uncertain. To allow for this uncertainty, structures and machines must be designed with a built in factor of safety

$$Factor\ of\ safety\ (F.S) = \frac{Failure\ load}{Max.allowable\ load} > 1$$

or

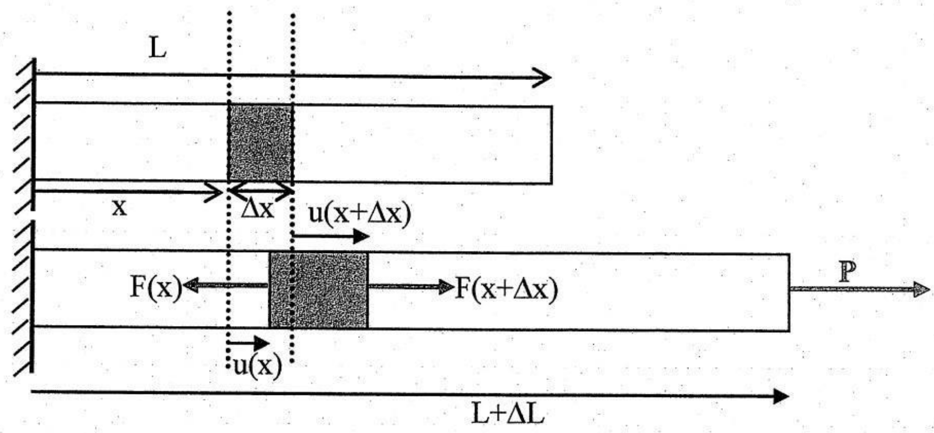
$$Factor\ of\ safety\ (F.S) = \frac{Failure\ or\ yield\ stress}{Max.allowable\ stress} > 1$$

We will deal with two kinds of problems:

1. Evaluation of a given structure (what is the max. allowable load on a structure or machine? What is the built in FS in the structure)
2. Design materials/dimensions of components with built in FS.

ME 323 Review

Axial deformation



Axial deformation assumptions (regardless of whether material is uniform or not)
1. Axis remains straight after the deformation, and
2. Cross section remains plane and perpendicular to the rod axis, and undergoes axial displacement.

If cross-section made of different materials with E_i, A_i, i=1..N, then

$$F(x) = \frac{du}{dx}\left(E_1A_1 + E_2A_2 + ...\right)$$

If uniform material in the cross-section then

$$F(x) = EA\frac{du}{dx} \quad or \quad \frac{du}{dx} = \frac{1}{EA}F(x)$$

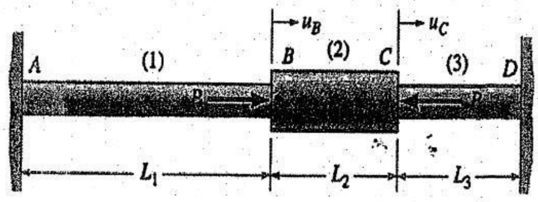
To find displacements u(x) given F(x), one simply needs to integrate

$$\mathrm{u}(\mathrm{x}_2) - \mathrm{u}(\mathrm{x}_1) = \frac{1}{EA}\int_{x_1}^{x_2} F(x)dx = e \text{ (elongation)}$$

Furthermore, if F(x) is constant between x_1 and x_2 then this simplifies more

$$\mathrm{u}(\mathrm{x}_2) - \mathrm{u}(\mathrm{x}_1) = \frac{F(x_2 - x_1)}{EA} = e \text{ (elongation)}$$

Sign convention for axial deformation formulas:
- *F>0 means tensile axial force resultant*
- u>0 means axial displacement towards +x direction

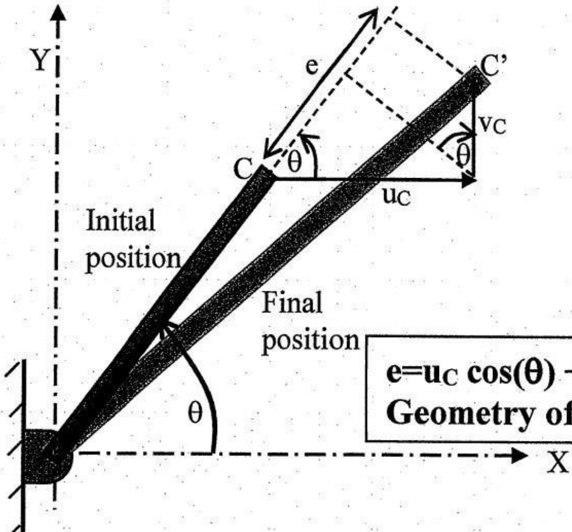


P3.5-10 and P3.9-2

- In many problems however, one cannot solve for the internal resultants from a free body diagram of a section. There is one additional constraint or support which adds more unknowns than the number of static equilibrium equations.

To solve *axial* deformation problems involving statically indeterminate structures, our strategy will be the following:

1. Make sections and FBD, and write down all the static equilibrium equations for the internal resultant forces (more unknowns than equations).
2. Write down force-displacement relationships for each part.
3. (MOST IMPORTANT) determine how the elongations of the different parts depend on each other (i.e. derive or record the *compatibility equation*)

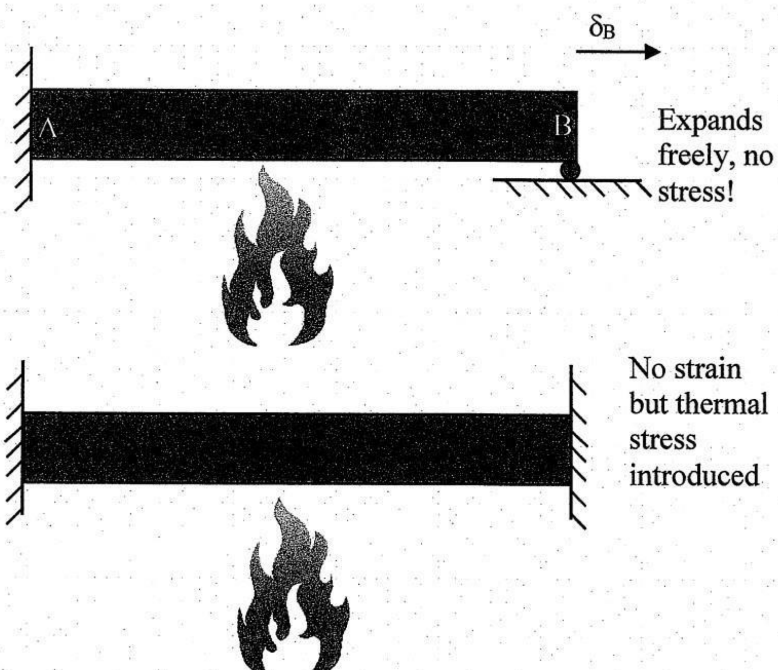


$e = u_C \cos(\theta) + v_C \sin(\theta)$
Geometry of deformation for truss element

Important sign conventions in using the geometry of deformation for truss element formula:

- θ measured CCW with respect to + X axis.
- X axis originates at the point where rod is pinned to ground!!!

Axial thermal stresses



Temperature increase or decrease causes additional stresses only if structure is statically indeterminate!

Torsion

$$\gamma(x,\rho) = \rho\frac{d\phi}{dx}$$
$$\tau = G\gamma$$ (Geometry of deformation, Hooke's law)

$$T(x) = \int_A \rho G\left(\rho\frac{d\phi}{dx}\right)dA = \frac{d\phi}{dx}\int_A G\rho^2 dA$$ **(Torque-twist equation)**

Simplifications

- If the rod is made of a uniform material over the cross section (G is constant over the section) then the torque-twist equation simplifies to
$$T(x) = G\frac{d\phi}{dx}\int_A \rho^2 dA = GI_p\frac{d\phi}{dx}$$
- Additionally, if BOTH the internal resultant torque T(x)=T is constant, AND if G is constant over the cross section, then between $x=x_1$ and $x=x_2$, $x_2 > x_1$, then by integrating the torque-twist equation above, we get
$$\phi(x_2) - \phi(x_1) = \frac{T(x_2 - x_1)}{GI_p} = \frac{TL}{GI_p}$$ **(Simplified torque-twist equation)**
- Finally we derived the all important torsion formula:

$$\tau_{x\theta}(x,\rho) = \rho\frac{T}{I_p}$$ **(Torsion formula)**

Sign conventions

1. Internal resultant torques are assumed positive if they are *in the right hand sense* about *the normal to the exposed section of the beam.*
2. The twist angle ϕ is positive when measured in the *right handed sense* with *respect to the +X axis.*
3. In the simplified torque-twist equation, $\phi(x_2) - \phi(x_1) = \frac{TL}{GI_p}$, where $x_2 > x_1$

ME 323 Review

Problem solving strategy (statically determinate problems)

1. *Calculate the internal torque resultants* along the length of the beam.
2. To calculate the twist angles, see if the *simplified torque twist* relation can be applied, if not use the *general torque twist* relation to work out twist angles along the length of the rod.
3. Calculate stresses using the *torsion formula.*

Statically indeterminate problems in torsion

Consider a straight beam with external torques applied to it at various points. In the statically determinate problems we have studied in the last two classes, the internal torque resultants are easily calculated by making sections and applying static equilibrium equations. In many problems, however, it is impossible to determine the internal torque resultants using sections and static equilibrium equations alone. Such problems are statically indeterminate, and feature more unknown internal torques than the number of the static moment balance equations.

General strategy for solving statically indeterminate torsion problems

1. Make sections and write out all the static moment balance equations for the internal resultant torques.
2. Write out the torque-twist relation (usually the STTE) for each part of the beam.
3. Look at the geometry of deformation and figure out the *compatibility* equations.
4. Substitute (2) in (3) and solve with (1) to yield the internal torques in each section of the beam.
5. Finally use the torsion formula to calculate the maximum stress in each part.

ME 323 Review

Shear force bending moment diagrams

How to calculate shear force (V-M) diagrams using equilibrium relationships

Geometric meaning of the equilibrium relationships for beams

- $$\frac{dV(x)}{dx} = p(x)$$
  The slope of the shear force diagram at any location x equals the value of the distributed eternal force at that 'x'.

- $$V(x_2) = V(x_1) + \int_{x_1}^{x_2} p(s)\,ds$$ (integral form)
  The shear force at point x_2 further down the beam from x_1 equals the area under the external loading curve between the two points.

- $$\frac{dM(x)}{dx} = V(x)$$
  The slope of the bending moment diagram at any location x equals the value of the shear force at that 'x'.

- $$M(x_2) = M(x_1) + \int_{x_1}^{x_2} V(s)\,ds$$ (integral form)
  The bending moment at a point x_2 further down the beam from x_1 equals the area under the shear force curve between the two points.

- $$\Delta V_A = V_{A+} - V_{A-} = P_0$$
  The shear force diagram has a step jump whenever an external point force is encountered. The value of the shear force jump equals the value of the external point force.

- $$\Delta M_A = M_{A+} - M_{A-} = -M_0$$
  The bending moment diagram has a step jump whenever an external point moment is encountered. The value of the bending moment jump equals the negative of the value of the external point moment.

ME 323 Review

General method for V-M diagrams:

Step 1: Calculate all support reactions from the FBD of the entire structure.

Support reactions count as "external forces and moments" for the equilibrium relationships!

Step 2: Apply the above equations to first sketch the V(x) and then M(x).

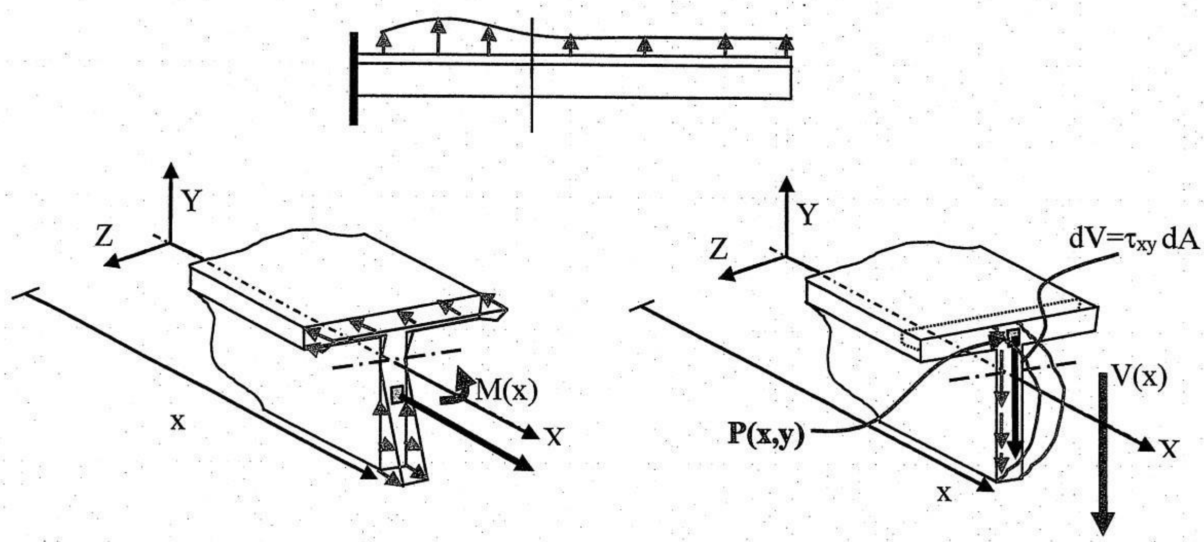
Don't forget the sign conventions for using equilibrium relationships!

- A positive internal shear force, V, acts in the –y direction on a +x face.
- A positive internal bending moment, M, makes the + y face of the beam concave.
- An axial force resultant, F, is positive if it is in tension.
- External forces are positive if they act in the +Y direction.
- External moments are positive if they act CCW with respect to the +Z axis.

Finally when sketching the V-M diagrams be sure to indicate clearly all the key values and slopes must be clearly listed on the diagrams - you will be graded on this in the exam. These include:

- *Values of slopes in all sections of each diagram*
- *Values of step jumps in diagrams*
- *Values of maximum value of shear force and bending moment in the diagram*
- *Values of 'x' where max. values of shear force and bending moment are reached*

Stresses in beams



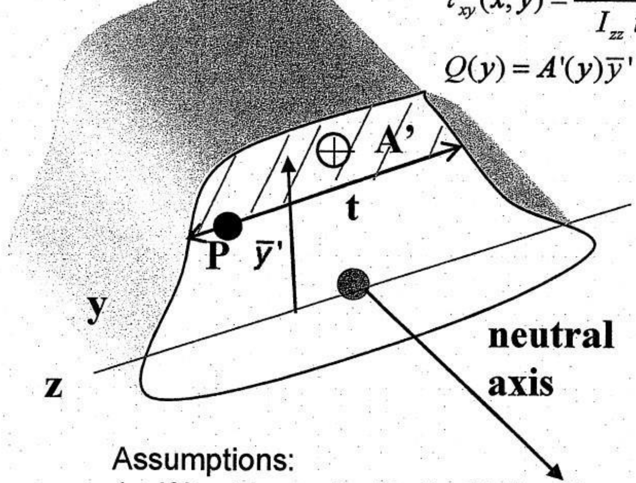
Internal bending moment leads to **flexural stresses** on the cross section

$$\sigma_{xx}(x,y) = \frac{-M(x)\,y}{I_{zz}}$$ The **Flexure Formula**

But internal shear resultant V(x) leads to a **shear stress** distribution on the cross section

$$\tau_{xy}(x,y) = \frac{V(x)\,Q(y)}{I_{zz}\,t(y)}$$ The **Shear-stress Formula**

$$Q(y) = A'(y)\,\bar{y}'$$

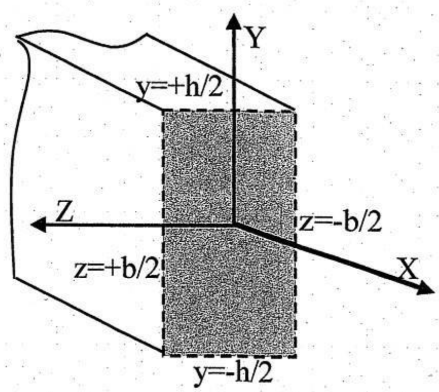


Assumptions:

1. If load is applied in the Y direction, shear stress τ_{xy} varies on the cross section in the Y direction and remains constant in the Z direction.
2. Other shear stresses τ_{xz} etc. are negligible.
3. Distribution of flexural stress is not affected by the presence of shear stress τ_{xy}.

The 2nd area moment about the z axis

- $\int_A y^2 dA = I_{zz}$, the 2nd area moment of the cross section about the Z axis, measured relative to the neutral (centroidal) axis.
- For <u>a rectangular cross section</u>



$$I_{zz} = \int_A y^2 dA$$

$$= \int_{z=-b/2}^{z=+b/2} \int_{y=-h/2}^{y=+h/2} y^2 dy \quad dz$$

$$= \int_{z=-b/2}^{z=+b/2} \left[\frac{y^3}{3} \right]_{-h/2}^{+h/2} dz = \frac{h^3}{12} \int_{z=-b/2}^{z=+b/2} 1.dz$$

$$I_{zz} = \frac{bh^3}{12}$$

- <u>For an annular cross section of outer diameter d_o, and inner diameter of d_i</u>

$$I_{zz} = \frac{\pi\left(d_o^4 - d_i^4\right)}{64}$$

- <u>For a circular cross section of outer diameter d_o</u>

$$I_{zz} = \frac{\pi d_o^4}{64}$$

- <u>For more complicated shapes, one needs to use the parallel axis theorem.</u>

$I_{z'z'} = I_{zz} + A(\mathrm{d})^2$

zz: axis passing through centroid

z'z' is axis parallel to zz but displaced by an amount 'd'

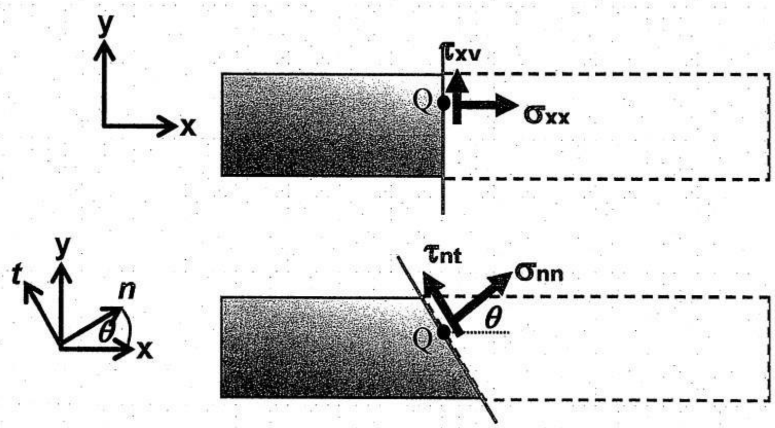
Stress transformation and the Mohr's circle

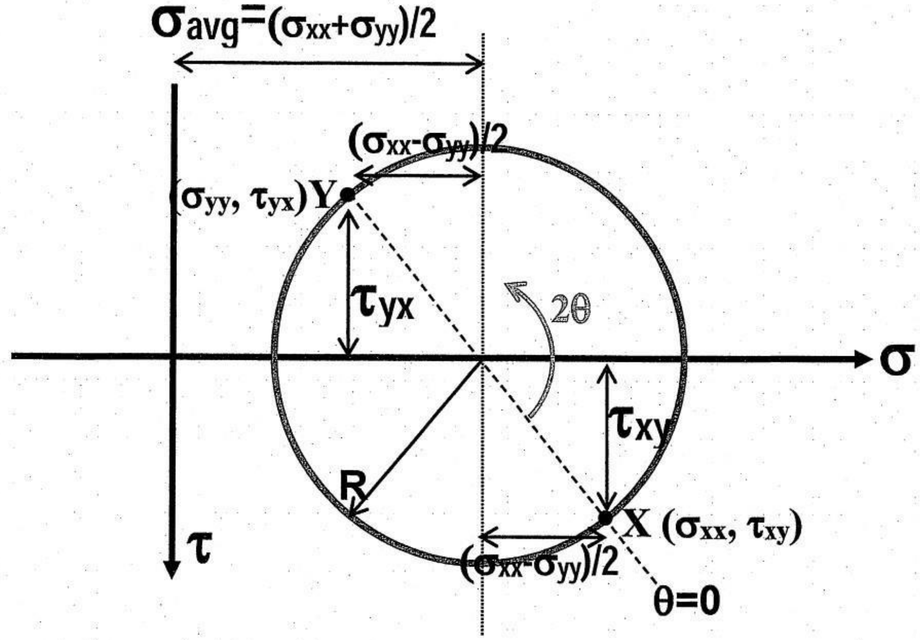
Stress transformation equations

$$\sigma_{nn}(\theta) = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{(\sigma_{xx} - \sigma_{yy})}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

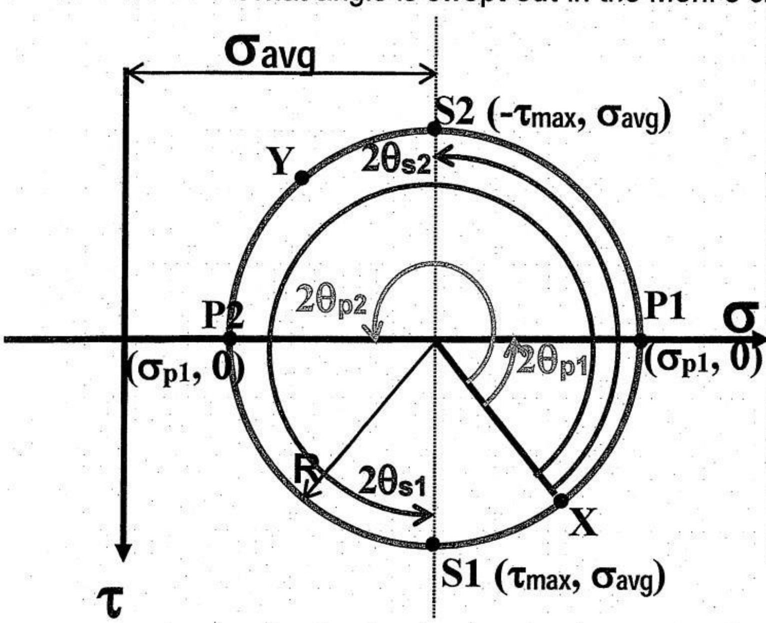
$$\tau_{nt}(\theta) = -\frac{(\sigma_{xx} - \sigma_{yy})}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

θ=0 is the physical X axis. The stress transformation equations tell us how (σ_{nn}, τ_{nt}) change as θ changes from 0.





- Where does the state of stress at $\theta = 0$ $\mathbf{X} = (\sigma_{xx}, \tau_{xy})$ lie on the Mohr's circle?
- Where does the state of stress at $\theta = 90^\circ$ $\mathbf{Y} = (\sigma_{yy}, \tau_{yx})$ lie on the Mohr's circle?
- What angle is swept out in the Mohr's circle?



Principal stresses and directions

$\sigma_{p1} = \sigma_{avg} + R$ found on section oriented at θ_{p1} wrt X axis

$\sigma_{p2} = \sigma_{avg} - R$ found on section oriented at θ_{p2} wrt X axis

The two principal directions are physically 90° apart

Maximum shear stresses and directions

$\tau_{\max} = R$ found on section oriented at θ_{s1} wrt X axis

$\tau_{\min} = -R$ found on section oriented at θ_{s2} wrt X axis

The two directions for max/min shear are physically 90° apart

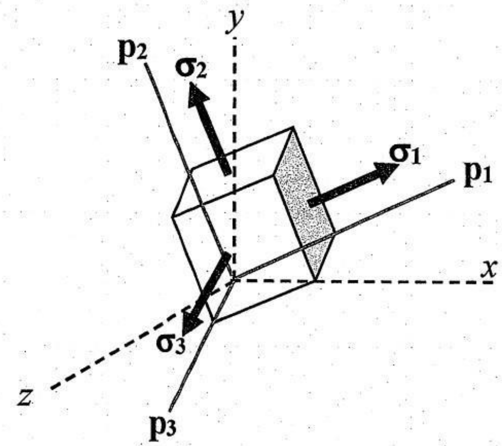
The direction for maximum / min shear is at 45° to a principal direction

Triaxial stress, principal directions, and absolute maximum shear stress

For a general three dimensional state of stress, it can be shown that: **there are three principal stresses, and the corresponding principal planes are mutually perpendicular**. There are no shear stresses on the principal planes and the three principal stresses are labeled in the order- maximum, intermediate, and minimum

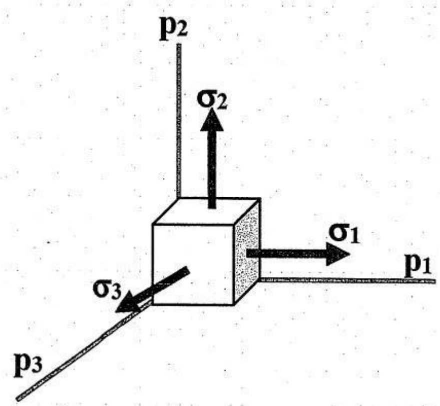
$\sigma_1 \equiv \sigma_{\min}, \sigma_2 \equiv \sigma_{\text{int}}, \sigma_3 \equiv \sigma_{\max}, \qquad \sigma_1 \geq \sigma_2 \geq \sigma_3$

Because all the faces are free of shear stress, this stress element is said to be in a state of triaxial stress.



Even for a plane stress situation, we need to determine the **absolute maximum shear stress**, the largest magnitude shear stress acting in any direction on any plane passing through that point. To understand this better let us assume that we know all three principal directions and principal stresses

$\sigma_1 \equiv \sigma_{\min}, \sigma_2 \equiv \sigma_{\text{int}}, \sigma_3 \equiv \sigma_{\max}, \qquad \sigma_1 \geq \sigma_2 \geq \sigma_3$



Clearly

$$(\tau_{\max})_{p1-p2} = \frac{\sigma_1 - \sigma_2}{2} \quad (\tau_{\max})_{p2-p3} = \frac{\sigma_2 - \sigma_3}{2} \quad (\tau_{\max})_{p3-p1} = \frac{\sigma_1 - \sigma_3}{2}$$

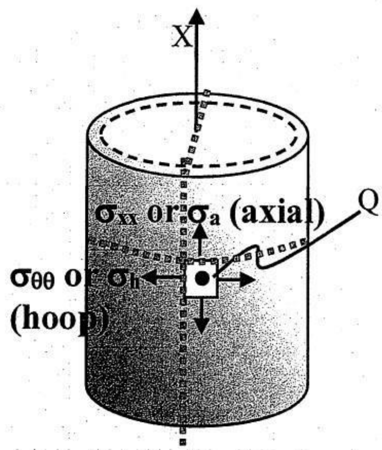
Therefore

$$(\tau_{\max})_{abs} = \frac{\sigma_{\max} - \sigma_{\min}}{2}$$

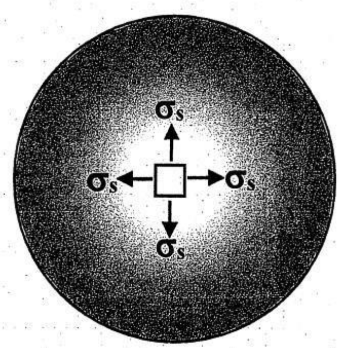
and this shear stress acts on a plane or cut section whose normal bisects the corresponding principal directions.

ME 323 Review

Pressure vessels and combined loading



$$\sigma_h = \frac{pr}{t}; \quad \sigma_a = \frac{pr}{2t}$$



$$\sigma_{sphere} = \frac{pr}{2t}$$

General procedure to calculate stresses due to combined loads

General procedure

1. Determine the internal resultants by making cut sections and drawing free body diagrams.
2. Calculate the stress on the cut section due to each internal resultant separately.
3. Combine individual stresses at specified points and calculate the state of stress at the specified points.

In problems dealing with combined loads, try not to use the signs in the stress formulas directly but rather work out the directions of the internal resultants, and infer which way the stresses should be pointing.

ME 323 Review

Beam bending (deflection curve)

$$(EIv'')' = V(x) \qquad \textbf{Shear Deflection Equation}$$

$$(EIv'')'' = p(x) \qquad \textbf{Load-Deflection Equation}$$

To solve for the deflection curve, we will use the following method:

2nd order integration method

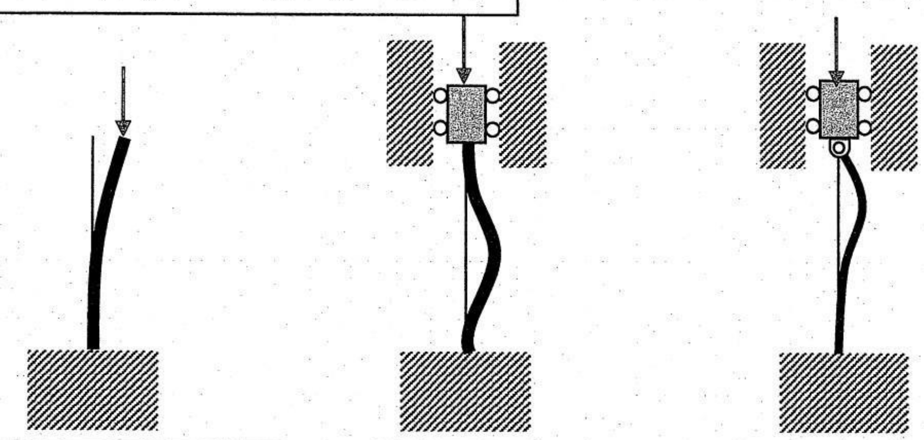
- Calculate the Shear-Force bending moment diagram, and write down expressions for M(x).
- Integrate the moment-curvature equation (2), with appropriate boundary and continuity equations to get v(x), the deflection curve.

For statically indeterminate problems, find the internal bending moment in terms of unknown support reactions. In the final deflection equation you will have additional boundary conditions since the problem is indeterminate. All together you will have as many equations as unknowns.

Beam buckling

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \qquad \sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}$$

where Le is the effective length of the column



Fixed-free $L_e = 2L$

Fixed-fixed $L_e = 0.5L$

Fixed-pinned $L_e = 0.7L$

ME 323 Review

Energy methods

Axial	$U=\frac{1}{2}\int_0^L \frac{F^2 dx}{EA}=\frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$
Torsion	$U=\frac{1}{2}\int_0^L \frac{T^2 dx}{GI_p}=\frac{1}{2}\int_0^L GI_p\left(\frac{d\phi}{dx}\right)^2 dx$
Bending-flexure	$U_\sigma=\frac{1}{2}\int_0^L \frac{M^2 dx}{EI}=\frac{1}{2}\int_0^L EI\left(\frac{d^2u}{dx^2}\right)^2 dx$
Bending-shear	$U_\tau=\frac{1}{2}\int_0^L \frac{f_s V^2 dx}{GA}$ $f_s=\int_A \frac{Q^2(x,y)}{t^2(x,y)}dA$

Work Energy principle for calculating deflections

If the stresses in the body do not exceed the elastic limit, all of the work done on a body by external forces is stored in the body as elastic strain energy.

$$W_{ext}=U$$

This idea can be used to calculate the deflections of beams in a very elegant and fast manner. *However the method is restricted to single loads, and does not quite work for multiple loads.*

Castigliano's 2nd theorem for statically determinate problems

If several loads P_i i=1..N and external moment M_i,i=1..Q act on a body, then the deflection Δ_i in the direction of the applied load can simply be calculated as follows:

$$\Delta_i = \frac{\partial U(P_1, P_2, \ldots P_N, M_1, M_2, \ldots M_Q)}{\partial P_i}$$

where the total potential (strain) energy is written in terms of the loads P_i and M_i. Furthermore the local rotation θ_i in the direction of the applied moment can simply be calculated as follows:

$$\theta_i = \frac{\partial U(P_1, P_2, \ldots P_N, M_1, M_2, \ldots M_Q)}{\partial M_i}$$

Castigliano's 2nd theorem for statically indeterminate problems

Suppose that in addition to the P_i i=1..N and external moments M_i,i=1..Q, that we have S redundant forces or moments, R_i. Recall that one interpretation of the redundant forces or moments is that they create some constraint – or prevent the motion of the structure in some ways. Accordingly, we will write that the deflections due to the redundant forces should be zero. We get

$$0 = \frac{\partial U(P_1, P_2, \ldots P_N; \quad R_1, R_2, \ldots\ldots R_{RN})}{R_i}$$

$$\Delta_i = \frac{\partial U(P_1, P_2, \ldots P_N; \quad R_1, R_2, \ldots\ldots R_{RN})}{\partial P_i}$$

So now we have N+S equations for as many variables. Typically the first equation above can be used to calculate the redundant forces.