→ ***See exam 1 and exam 2 study guides for previous materials covered in exam 1 and 2.***

Stress transformation

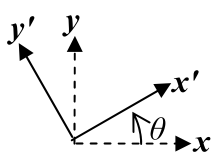
In summary, the stress transformation equations are:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

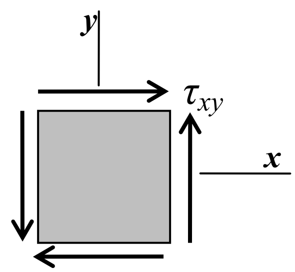
$$\tau_{x'y'} = \frac{-\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

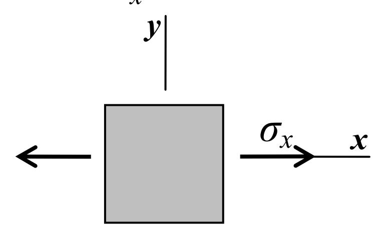
The angle θ is the angle measured counterclockwise from x to x':



Positive τ_{xy}:



Positive σ_x:



Important:

Principal Plane: The maximum and minimum normal stresses are called the "*principal stresses*".

The orientation angle of this plane, θ_p, is:

$$\tan\left(2\theta_p\right) = \frac{\tau_{xy}}{\left(\frac{\left(\sigma_x - \sigma_y\right)}{2}\right)}$$

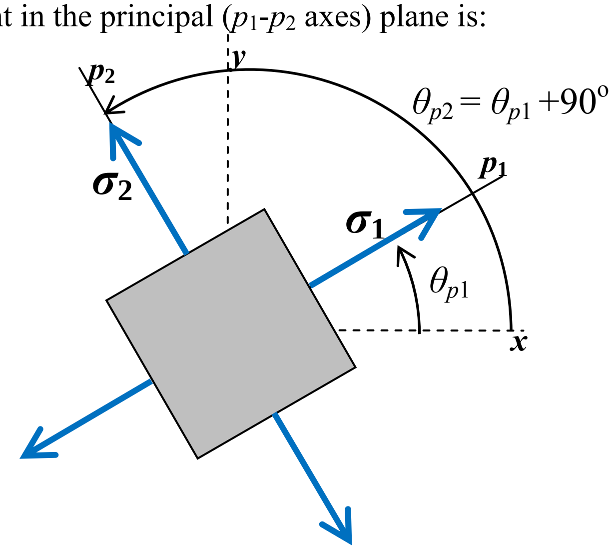
**See next page on how to use this eqn*

The stresses in this plane are:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$\tau_p = 0$$

Therefore, the stress element in the principal (p_1-p_2 axes) plane is:



Maximum in plane shear stress

The orientation angle of this plane, θs, is:

$$\tan\left(2\theta_s\right) = \frac{\left(\frac{-\left(\sigma_x - \sigma_y\right)}{2}\right)}{\tau_{xy}}$$

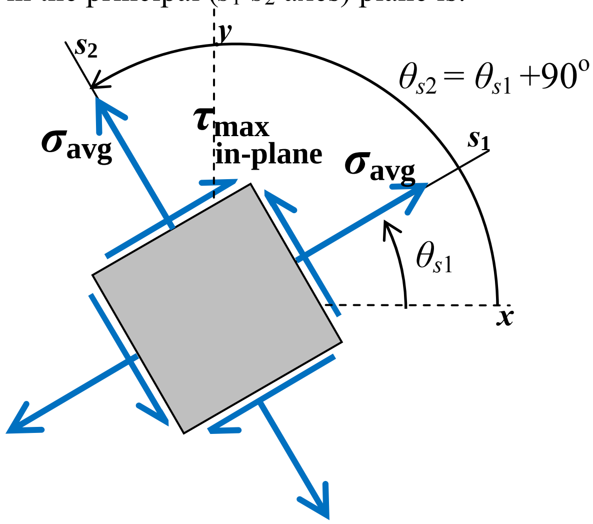
**See next page on how to use this eqn*

The stresses in this plane are:

$$\tau_{\substack{\max \\ in-plane}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

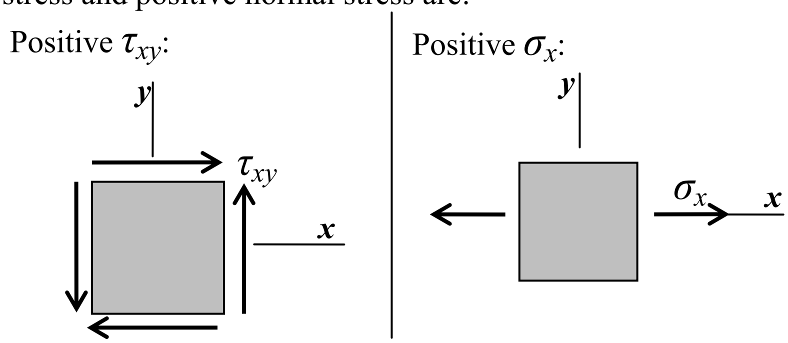
$$\sigma_{s1} = \sigma_{s2} = \sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

Therefore, the stress element in the principal (s_1-s_2 axes) plane is:



Important points:

1. θ_{p1} is measured CCW from x to p_1
2. θ_{s1} is measured CCW from x to s_1
3. positive shear stress and positive normal stress are:



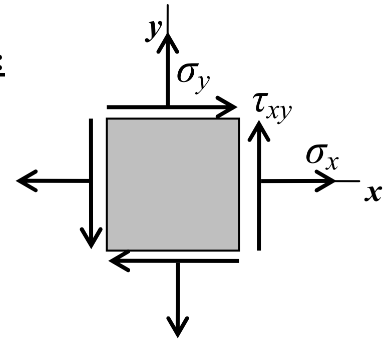
4. When calculating the angle from $\tan\left(2\theta_p\right) = \frac{\tau_{xy}}{\left(\frac{\left(\sigma_x - \sigma_y\right)}{2}\right)}$, we can get two angles, but we don't know which is θ_{p1} which is θ_{p2}. To find out, here is what we can do:

➔ Calculate $\sigma_{x'}$ using one of the angles. If $\sigma_{x'} = \sigma_1$, then that angle is θ_{p1}, if $\sigma_{x'} = \sigma_2$, then that angle is θ_{p2}. (See example 9.3 on page 448 of textbook.)

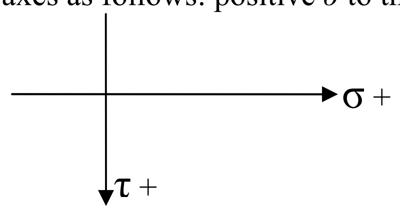
Mohr's Circle

Mohr's circle is a graphical representation of the stress transformation equations.

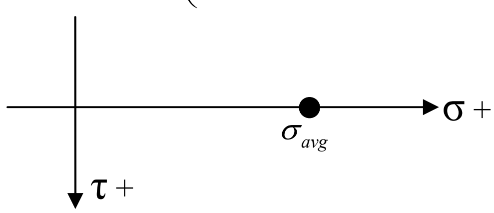
Procedure to construct a Mohr's Circle, given a stress state:



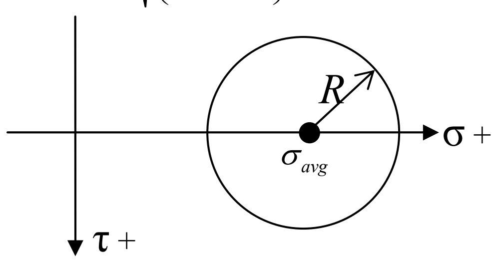
1. Draw the σ and τ axes as follows: positive σ to the right and positive τ down:



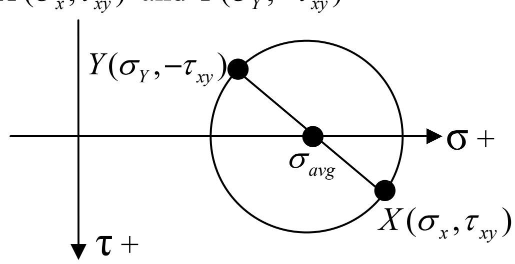
2. Position the center of the circle at $\left(\sigma = \sigma_{avg} = \dfrac{\sigma_x + \sigma_y}{2}; \quad \tau = 0 \right)$



3. The radius of the circle is $R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^{\ 2}}$

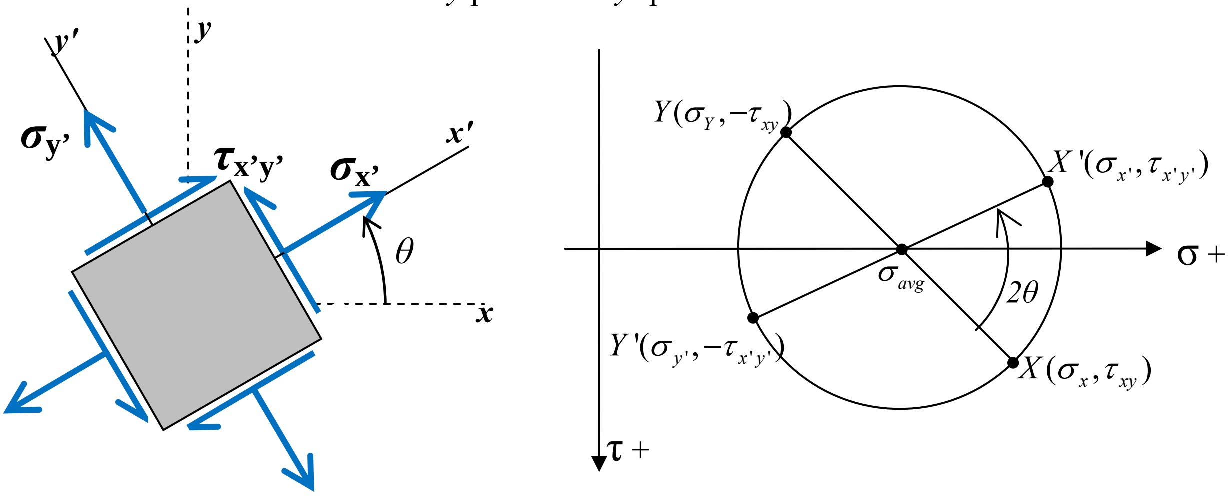


4. The stress state in the x-y plane shown above are represented by a straight line connecting two points on the Mohr's Circle: $X(\sigma_x, \tau_{xy})$ and $Y(\sigma_Y, -\tau_{xy})$



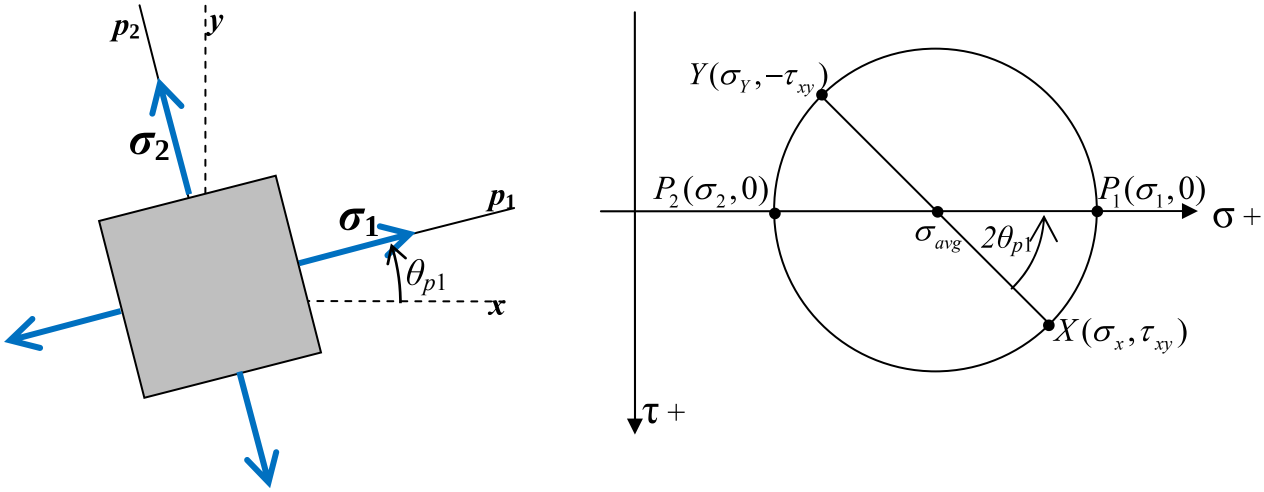
<u>How to use the Mohr's Circle</u>

1. Stress transformation from *x*-*y* plane to *x*'-*y*' plane in a Mohr's Circle:

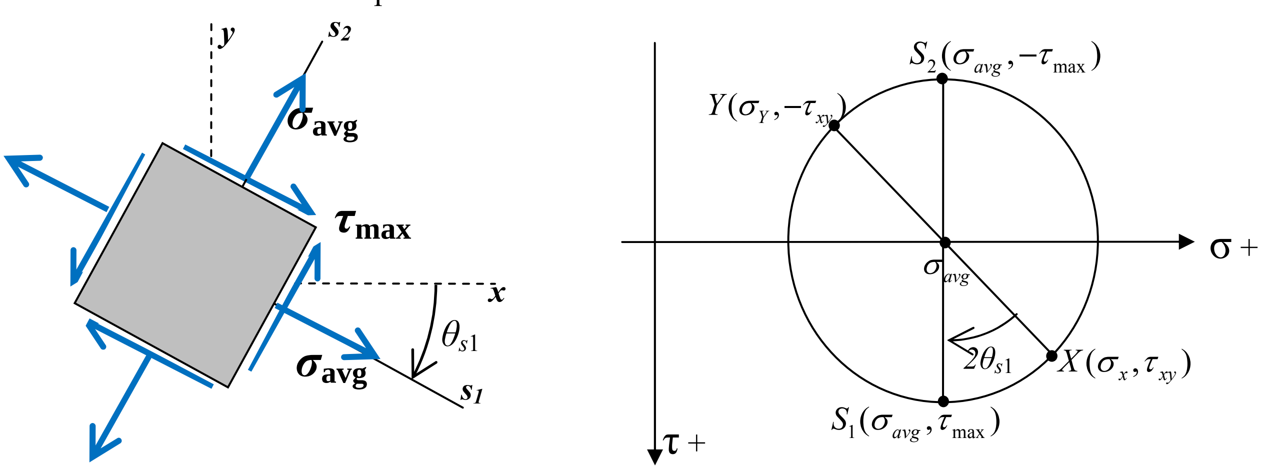


***θ* CCW rotation (*x* to *x'*) in stress element = 2*θ* CCW rotation in Mohr's Circle (*X* to *X'*)**

2. Plane of principal stress



3. Plane of maximum in-plane shear stress



Failure Theories

Understand how to use failure theories

→ e.g. finding minimum cross sectional area, finding maximum weight that a structure can support, etc.

→using failure theories with factor of safety

Theories of failure for **brittle** materials,

1. ***Maximum Normal Stress Theory***

   This failure theory *assumes that the ultimate stress of the material in tension and compression are equal*.

   → Failure occurs when the *maximum normal stress (principal stress)* in the material reaches a value that is equal to the *ultimate normal stress*.

$$|\sigma_1| = \sigma_u$$
$$|\sigma_2| = \sigma_u$$

2. ***Mohr's Failure Criterion***

   This failure theory is for brittle materials whose ultimate strength in tension and compression are different.

In tension, $\sigma_{\max,tension} = \sigma_{U,tension}$

In compression, $|\sigma_{\max,compression}| = \sigma_{U,compression}$

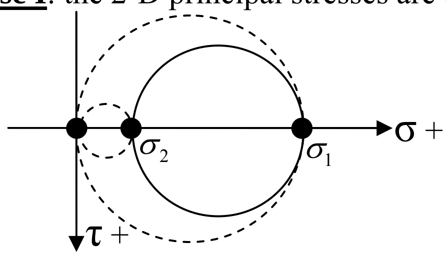
Theories of failure for **<u>ductile</u>** materials,

1. <u>Maximum Shear Stress Theory (Tresca yield criterion)</u>

→ Failure occurs when the *<u>absolute maximum shear stress</u>* in the material is equal to the *<u>shear stress that causes the material to yield in uniaxial tension test</u>*.

$$\tau_{\underset{\max}{abs}} = \frac{\sigma_Y}{2}$$

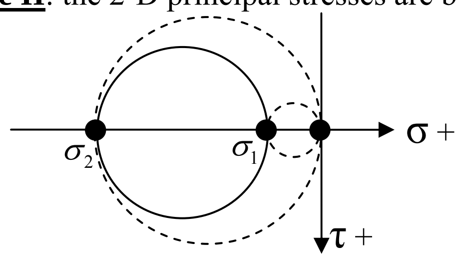
<u>Case I</u>: the 2-D principal stresses are both positive



Absolute maximum shear stress = largest radius

$$\tau_{\underset{\max}{abs}} = \frac{|\sigma_1|}{2}$$

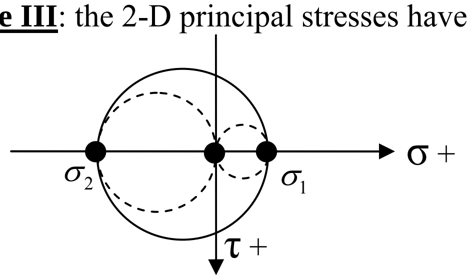
<u>Case II</u>: the 2-D principal stresses are both negative



Absolute maximum shear stress = largest radius

$$\tau_{\underset{\max}{abs}} = \frac{|\sigma_2|}{2}$$

<u>Case III</u>: the 2-D principal stresses have opposite signs



Absolute maximum shear stress = largest radius

$$\tau_{\underset{\max}{abs}} = \frac{|\sigma_1 - \sigma_2|}{2}$$

Therefore, in terms of the principal stresses,

$$\left.\begin{array}{l} |\sigma_1| = \sigma_Y \\ |\sigma_2| = \sigma_Y \end{array}\right\} \text{ if } \sigma_1 \text{ and } \sigma_2 \text{ have same signs}$$

$$|\sigma_1 - \sigma_2| = \sigma_Y \quad \text{ if } \sigma_1 \text{ and } \sigma_2 \text{ have opposite signs}$$

2. <u>Maximum Distortion Energy Theory</u>

This failure theory uses the *<u>maximum distortion energy</u>* (the energy required to *change the shape of the material without changing the volume*) to characterize failure.

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2$$

Beam Deflection

Beam deflection: 2nd order integration method

1. *Determine the internal bending moment equation* ***for each continuous segment***
   a. Find the reaction forces
   b. Derive the internal bending moment through equilibrium method
   Note: It is also possible to use the graphical method. **However**, it is important to remember that the actual equation of *M* must be derived.

2. *Moment-Deflection equation* ***for each continuous segment****t*
   Integrate $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{M}{EI}}$ to find the deflection equation

3. *Boundary Conditions & Continuity Equations*
   Use the boundary conditions and continuity equations to find the integration constants in the deflection equations found in step 2.

Boundary Conditions:

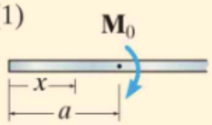
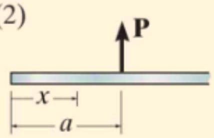
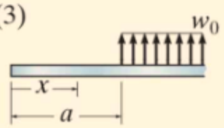
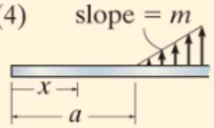
Roller	Zero deflection $u = 0$	No deflection slope restriction
Pin	Zero deflection $u = 0$	No deflection slope restriction
Fixed end	Zero deflection $u = 0$	Zero deflection slope $\frac{du}{dx} = 0$

Continuity Equations:
For each continuous section, we have different internal bending moment equation. Consequently, the deflection equations are different. To ensure that the different equations result in a continuous deflection shape, we will enforce continuity equations. At each discontinuity point,

$$u_1 = u_2$$
$$\frac{du_1}{dx_1} = \frac{du_2}{dx_2}$$

Statically determinate vs. statically indeterminate

From Table 12-2 of textbook:

Loading	Loading Function $w = w(x)$	Shear $V = \int w(x)dx$	Moment $M = \int V dx$
(1) 	$w = M_0\langle x-a\rangle^{-2}$	$V = M_0\langle x-a\rangle^{-1}$	$M = M_0\langle x-a\rangle^{0}$
(2) 	$w = P\langle x-a\rangle^{-1}$	$V = P\langle x-a\rangle^{0}$	$M = P\langle x-a\rangle^{1}$
(3) 	$w = w_0\langle x-a\rangle^{0}$	$V = w_0\langle x-a\rangle^{1}$	$M = \frac{w_0}{2}\langle x-a\rangle^{2}$
(4) 	$w = m\langle x-a\rangle^{1}$	$V = \frac{m}{2}\langle x-a\rangle^{2}$	$M = \frac{m}{6}\langle x-a\rangle^{3}$

<u>Discontinuity function</u>

1. Calculate support reactions
2. Use the discontinuity functions in the table above to express M(x) as a function of x.
   <u>Note</u>: Since we are expressing the equation using discontinuity function, only 1 equation is needed for any beam.
3. Use the moment-displacement equation $EI\frac{d^2u}{dx^2} = M$ and integrate to get u(x)
4. Use boundary conditions to find integration constants (continuity *not* needed for this method)

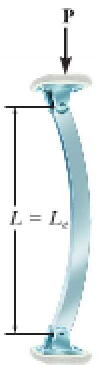
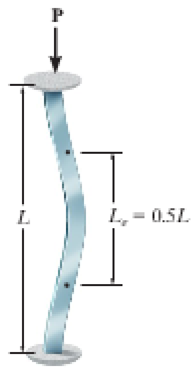
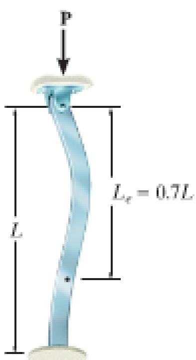
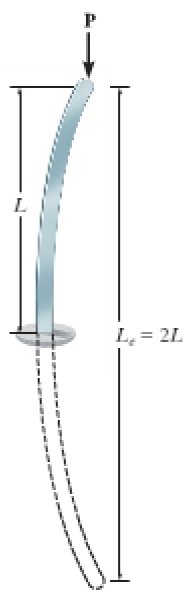
Buckling

Buckling can happen when the internal load > P_{cr}.

For various types of support, the critical force equation for buckling becomes,

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$ ------------------------------------ *in general*

where $L_e = KL$ is the effective length of the column and K is listed in the table below,

<table>
<tr>
<td>Pinned-Pinned ends: $K = 1$

</td>
<td>Fixed-Fixed ends: $K = 0.5$

</td>
</tr>
<tr>
<td>Pinned-Fixed ends: $K = 0.7$

</td>
<td>Fixed-Free ends: $K = 2$

</td>
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</table>

Failure analysis: buckling vs. yielding (crushing)

Energy Methods
Work-Energy in deformable material:

Work done to deform an elastic material = Total strain energy (stored energy) in structure $W_e = U_i$

Work done to deform an elastic material

By a Force	$W_e = \frac{1}{2} F d$	F = Force applied at point **P** d = Deformation of point **P** in the direction of the force
By a Moment	$W_e = \frac{1}{2} M \theta$	M = Moment applied at point **P** θ = Deformation angle/slope at point **P** in the direction of the moment

Components of strain energy in the structure:

By axial loading	$U_{i,N} = \frac{N^2 L}{2AE}$	N = internal normal force
By torsion	$U_{i,T} = \frac{T^2 L}{2JG}$	T = internal torque
Bending		
By bending moment	$U_{i,M} = \int_L \frac{M^2}{2EI} dx$	M = internal bending moment
By transverse shear force	$U_{i,V} = \int_L \frac{f_s V^2}{2GA} dx$	V = internal shear force f_s = form factor = $\frac{A}{I^2} \int_A \left(\frac{Q^2}{t^2} \right) dA$ For a rectangular cross section, $f_s = \frac{6}{5}$

Total strain energy of the structure:

$$U_i = \frac{N^2 L}{2AE} + \frac{T^2 L}{2JG} + \int_L \frac{M^2}{2EI} dx + \int_L \frac{f_s V^2}{2GA} dx$$

Note: The contribution of strain energy from transverse shear force is negligible for long slender beams, therefore it can be neglected.

Impact loading: use the equation $KE_1 + PE_1 + W_e = KE_2 + PE_2$

For purely elastic deformation, strain energy is conserved. Therefore, strain energy can be included in potential energy.

Limitation of the Work-Energy principle: Only displacement in the direction of a single applied load can be computed.

To get around this limitation, we will use a more powerful method:

- Principle of Virtual Work (Ch. 14.5 in textbook)

Principle of Virtual Work

For trusses:
To find a displacement at point **P** (Δ_P), replace real loadings with a virtual load of magnitude 1 at point **P**, in the direction of the displacement.

$$\Delta_P = \sum_i \left(\frac{F_{virtual} F_{real} L}{AE} \right)_i = \sum_i \left(\frac{nNL}{AE} \right)_i$$

where,

i = truss i

$F_{virtual} = n$ = internal normal force of truss i for the case of virtual loading

$F_{real} = N$ = internal normal force of truss i for the case of real loading

For beams:
To find a displacement at point **P** (Δ_P), replace real loadings with a virtual load of magnitude 1 at point **P**, in the direction of the displacement.

$$\Delta_P = \int_L \frac{M_{virtual} M_{real}}{EI} dx = \int_L \frac{mM}{EI} dx$$

where,

$M_{virtual} = m$ = internal bending moment of truss i for the case of virtual loading

$M_{real} = M$ = internal bending moment of truss i for the case of real loading

Note: If angle at point **P** is needed, apply a virtual point moment of magnitude 1 at point **P**.

Strategy:

1. *Replace* real loadings with a virtual load (or moment) of magnitude 1 at the point where displacement is to be computed (in the same direction as the displacement).
2. Determine internal resultants for both virtual loading case and real loading case.
3. Apply the equation, $\Delta_P = \sum_i \left(\frac{nNL}{AE} \right)_i$ for trusses

   $\Delta_P = \int_L \frac{mM}{EI} dx$ for beams (transverse shear negligible)